richardwu.ca

MATH 239 Course Notes

Introduction to Combinatorics

Kevin Purbhoo • Spring 2017 • University of Waterloo

Last Revision: March 4, 2018

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. These notes are my interpretation and transcription of the content covered in lectures. The instructor has not verified or confirmed the accuracy of these notes, and any discrepancies, misunderstandings, typos, etc. as these notes relate to course's content is not the responsibility of the instructor. If you spot any errors or would like to contribute, please contact me directly.

1 May 3, 2017

1.1 Division is sketchy

Example 1.1. How many **different** outcomes are there from rolling 2 dices?
If dices are distinguishable:

$$6 \times 6 = 36$$

If dices are indistinguishable:

$$\frac{6 \times 6}{2!} + \frac{6}{3} = 21$$

Note that when we create our tuples, $(1,2)$ and $(2,1)$ tuples are reduced to one element. We *can* divide by 2!, **BUT** note that (k,k) tuples are mistakenly reduced by a factor of 2, thus we must add half of them back.
Division is sketchy!

1.2 A^0

Note the **Cartesian Power** is defined as

$$A^k = A \times A \times ... \times A = \{(a_1, ..., a_k | a_1, ..., a_k \in A\}$$

What is A^0?

$$A^0 = \{()\}$$

() is the empty list/tuple.
By extension

$$|A^k| = |A|^k$$

And so $|A^0| = |A|^0 = 1$.

2 May 8, 2017

2.1 Generating functions

Abstracts all *configurations* (and thus countable) into a function-like representation.
Let S be a set of "configurations".
Let $w : S \to \mathbb{N}$ be a "weighted function" which assigns each configuration $\sigma \in S$ a "weight" $w(\sigma) \in \mathbb{N}$.
Note $N = \{0, 1, 2, \ldots\}$.

Definition 2.1. The **generating series** for S with respect to w is

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

Example 2.1. Suppose S is a the set of binary strings of length 4 and $w : S \to N$ is the weight function.

$$w(\sigma) = \# \text{ of 1s in } \sigma$$

How many binary strings of length 4 have n 1s?

$\sigma \in S$	$w(\sigma)$	$x^{w(\sigma)}$
0000	0	1
0001	1	x^1
0010	1	x^1
0100	1	x^1
1000	1	x^1
0011	2	x^2
0110	2	x^2
$\vdots$	$\vdots$	$\vdots$

So our generating series would be

$$1 + 4x + 6x^2 + \ldots + x^4$$

Note the coefficients are the count of each configuration $x^{w(\sigma)}$ or each weight $w(\sigma)$.

3 May 10, 2017

Continuing from before...

Example 3.1. Another way to get $\Phi_S(x)$

$$\begin{aligned}\Phi_S(x) &= \sum_{n\geq 0} \binom{4}{n} x^n \\ &= \binom{4}{0}x^0 + \binom{4}{1}x^1 + \ldots + \binom{4}{4}x^4 + \binom{4}{5}x^5 + \ldots \\ &= \binom{4}{0}x^0 + \binom{4}{1}x^1 + \ldots + \binom{4}{4}x^4\end{aligned}$$

3.1 Binomial Thoerem

We can also use the **Binomial Theorem**

$$(1+x)^m = \sum_{k\geq 0}^{m} \binom{m}{k} x^k$$

So

$$\Phi_S(x) = (1+x)^4$$

How can we find the number of elements in S? **Plug in** $x = 1$

$$|S| = \Phi_S(1)$$

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$
$$= \sum_{\sigma \in S} 1$$

What about the sum of weights in S?

$$\Phi'_S(1)$$

In this example

$$\Phi'_S(x) = 1 \cdot 0x^{-1} + 4 \cdot 1x^0 + 6 \cdot 2x^1 + 4 \cdot 3x^2 + 1 \cdot 4x^3$$
$$= 0 + 4 + 12 + 12 + 4 = 32$$

See **Theorem 1.6.3.** for more details.

Example 3.2. Let S be the set of all binary strings

$$S = \{\epsilon, 0, 1, 00, 01, 10, 11, \ldots\}$$

where ϵ is the empty string.
Define the *weight function* $w : S \to \mathbb{N}$ where $w(\sigma)$ is the length of σ.
Counting problem How many binary strings of length n? **Answer** is 2^n.
What is $\Phi_S(x)$?

$$\Phi_S(x) = \sum_{n \geq 0} 2^n x^n$$
$$= \sum_{n \geq 0} (2x)^n$$
$$= \frac{1}{1 - 2x}$$

Note this is an *infinite geometric series* where $|2x| < 1$

3.2 Coefficient Notation

If we have a series $A(x) = \sum_{n \geq 0} a_n x^n$, then we can write

$$[x^n]A(x) = a_n$$

where $[x^n]A(x)$ means the **coefficient of** x^n **in** $A(x)$ or the **number of elements of** S **where** *weight* $= n$.

Example 3.3. By Binomial theorem

$$(1 + x)^m = \sum_{n \geq 0}^{m} x^k$$
$$= \sum_{n \geq 0}^{\infty} x^k$$

such that

$$[x^n](1+x)^m = \binom{m}{n}$$

3.3 *Formal* Power Series (FPS)

Power series are

$$A(x) = \sum_{x \geq 0} a_n x^n$$

What does **formal** mean? Use only algebraic manipulation.

Example 3.4.

$$\begin{aligned} 2+2 &= 4 && \textit{good} \\ (1+x)(1-x) &= 1-x^2 && \textit{good} \\ \frac{1-x^2}{1-x} &= 1+x && * \end{aligned}$$

* *Bad* if you plug in $x = 1$ such that LHS is undefined. *Good* if you treat division as the inverse of multiplication. In the *good* case, this is the **formal interpretation**.

Formal Two expressions are equal if you can get from one to the other using algebraic manipulations.

Analytic Two expressions are equal if they are equal when you plug in numbers.

We will **NEVER** plug in numbers and use the formal interpretations.

4 May 12, 2017

4.1 Formal Power Series (*cont.*)

What **algebraic manipulations** are we allowed?

Allowed
- collecting like terms
- distribution laws allow standard alg. laws

NOT allowed
- taking limits

4.2 Adding FPS

Simple example

Example 4.1.

$$(\sum_{n \geq 0} a_n x^n) + (\sum_{n \geq 0} b_n x^n) = \sum_{n \geq 0} (a_n + b_n) x^n$$

Breaking up summations

Example 4.2.

$$
\begin{aligned}
&(\sum_{n\geq 0} nx^n) + (\sum_{n\geq 0} \frac{1}{n+1}x^{n+1}) \\
&= (0x^0 + \sum_{n\geq 1} nx^n) + (\sum_{m\geq 1} \frac{1}{m}x^m) \\
&= \sum_{m\geq 1} (m + \frac{1}{m})x^m
\end{aligned}
$$

4.3 Multiplying FPS

Example 4.3.

$$
\begin{aligned}
&(\sum_{n\geq 0} a_n x^n)(\sum_{n\geq 0} b_n x^n) \\
&= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots \\
&= \sum_{n\geq 0} (\sum_{k\geq 0}^{n} a_k b_{n-k})x^n
\end{aligned}
$$

It's a good idea to modify the "dummy variable" in each summation if more complex manipulation is necessary. Using the previous example to illustrate

Example 4.4.

$$
\begin{aligned}
&(\sum_{n\geq 0} a_n x^n)(\sum_{n\geq 0} b_n x^n) \\
&= (\sum_{k\geq 0} a_k x^k)(\sum_{l\geq 0} b_l x^l) \\
&= \sum_{l\geq 0} (\sum_{k\geq 0} a_k x^k) b_l x^l \\
&= \sum_{l\geq 0} (\sum_{k\geq 0} a_k x^k b_l x^l) \\
&= \sum_{l\geq 0} \sum_{k\geq 0} a_k x^k b_l x^l
\end{aligned}
$$

Now let's substitute $n = k + l \rightarrow l = n - k$, where $n \geq 0$ from our domain of k and l

$$
\begin{aligned}
&= \sum_{n\geq 0} \sum k \geq 0^n a_k b_{n-k} x^n \\
&= \sum_{n\geq 0} (\sum k \geq 0^n a_k b_{n-k}) x^n
\end{aligned}
$$

4.4 Composite FPS

Summations may also be expressed with a coefficient

Example 4.5.

$$\begin{aligned}
&(1+3x)\sum_{n\geq 0} nx^n \\
&= (1\sum_{n\geq 0} nx^n) + (3x\sum_{n\geq 0} nx^n) \\
&= (\sum_{n\geq 0} nx^n) + (\sum_{n\geq 0}(3x)nx^n) \\
&= (\sum_{n\geq 0} nx^n) + (\sum_{n\geq 0} 3nx^{n+1}) \\
&\ldots
\end{aligned}$$

4.5 Algebraic Proofs with FPS

Example 4.6. How do we justify (without using infinite geometric series formula)

$$\sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}$$

We can write $A(x)B(x) = 1 \to A(x) = \frac{1}{B(x)}$

$$\begin{aligned}
(1-2x)\sum_{n\geq 0} 2^n x^n) &= (\sum_{n\geq 0} 2^n x^n) - (2x\sum_{n\geq 0} 2^n x^n) \\
&= \sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} 2^{n+1} x^{n+1} \\
&= \sum_{m\geq 0} 2^m x^m - \sum_{m\geq 1} 2^m x^m \\
&= (1 + \sum_{m\geq 1} 2^m x^m) - \sum_{m\geq 1} 2^m x^m \\
&= 1
\end{aligned}$$

Note there is no concept of "radius of convergence" since these are simply algebraic manipulations (formal).

Using the same logic, we may derive the general form

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n$$

Example 4.7. We can then deduce the previous example using substitution $x = 2x$

$$\frac{1}{1-2x} = \sum_{n\geq 0} (2x)^n = \sum_{n\geq 0} 2^n x^n$$

Example 4.8. There is however also incorrect ways to do this (**THIS IS WRONG!**). That is let $x = x - \frac{1}{2}$

$$\begin{aligned}\frac{1}{\frac{3}{2} - x} = \frac{1}{1 - (x - \frac{1}{2})} &= \sum_{n \geq 0} (x - \frac{1}{2})^n \\ &= \sum_{n \geq 0} \sum_{k \geq 0} x^k (-\frac{1}{2})^{n-k} \binom{n}{k} \\ &= \sum_{k \geq 0} \sum_{n \geq 0} x^k (-\frac{1}{2})^{n-k} \binom{n}{k} \\ &= \sum_{k \geq 0} (\sum_{n \geq 0} (-\frac{1}{2})^{n-k} \binom{n}{k}) x^k\end{aligned}$$

Notice that the nested summation is an infinite sum which requires taking a **limit**.
The **right way is to do**

$$\begin{aligned}\frac{1}{\frac{3}{2} - x} = \frac{2}{3}(\frac{1}{1 - \frac{2}{3}x}) &= \frac{2}{3} \sum_{n \geq 0} (\frac{2}{3}x)^n \\ &= \frac{2}{3} \sum_{n \geq 0} (\frac{2}{3})^n x^n \\ &= \sum_{n \geq 0} (\frac{2}{3})^{n+1} x^n\end{aligned}$$

5 May 15, 2017

5.1 Variations of Binomial Theorem

- $$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad n \in \mathbb{N}$$

- $$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- $$(1 - x)^{-n} = \sum_{k=0}^{n} \binom{n + k - 1}{k} x^k$$

  this is the **Negative Binomial Theorem**

5.2 Computing Coefficients

Example 5.1. Compute

$$[x^n](2 + x^2 + x^3)^m$$

The solution is

$$\begin{aligned}(2+x^2+x^3)^m &= 2^m(1+\frac{1}{2}x^2+\frac{1}{2}x^3)^m \\ &= 2^m\sum_{k\geq 0}\binom{m}{k}(\frac{1}{2}x^2+\frac{1}{2}x^3)^k \\ &= 2^m\sum_{k\geq 0}\binom{m}{k}(\frac{1}{2}x^2)^k(1+x)^k \\ &= 2^m\sum_{k\geq 0}\binom{m}{k}2^{-k}x^{2k}\sum_{j\geq 0}\binom{k}{j}x^j \\ &= \sum_{k\geq 0}\sum_{j\geq 0}\binom{m}{k}\binom{k}{j}2^{m-k}x^{2k+j}\end{aligned}$$

Let $n = 2k+j$ or $j = n-2k$

$$\begin{aligned} &= \sum_{n\geq 0}(\sum_{k\geq 0}^{\lfloor \frac{n}{2}\rfloor}2^{m-k}\binom{m}{k}\binom{k}{n-2k})x^n \\ \therefore [x^n](2+x^2+x^3)^m &= \sum_{k\geq 0}^{\lfloor \frac{n}{2}\rfloor}2^{m-k}\binom{m}{k}\binom{k}{n-2k}\end{aligned}$$

5.3 Sum and Product Lemmas

Sum Lemma Suppose $S = A\cup B$ where $A\cap B = \varnothing$ and we have a weight function on S. Then

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

Proof.

$$\begin{aligned}\Phi_S(x) &= \sum_{\sigma\in S}x^{w(\sigma)} \\ &= \sum_{\sigma\in A}x^{w(\sigma)} + \sum_{\sigma\in B}x^{w(\sigma)} \\ &= \Phi_A(x)+\Phi_B(x)\end{aligned}$$

□

Product Lemma

Example 5.2. You have 5 loonies and 4 toonies.

(a) How many ways to make \$9?

(b) \$10?

Let S be the set of all coins where A is the set of loonies and B the set of toonies.

# of loonies	0	1	2	3	4	5
Toonie weights	x^0	x^1	x^2	x^3	x^4	x^5
$0 \to x^0$	x^0	x^1	x^2	x^3	x^4	x^5
$1 \to x^2$	x^2	x^3	x^4	x^5	x^6	x^7
$2 \to x^4$	x^4	x^5	x^6	x^7	x^8	x^9
$3 \to x^6$	x^6	x^7	x^8	x^9	x^{10}	x^{11}
$4 \to x^8$	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}

$$\begin{aligned}\therefore \Phi_S(x) &= x^0 + x^1 + x^2 + \ldots + x^{13} \\ &= x^0 + x^1 + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 3x^7 + 3x^8 + 3x^9 + 2x^{10} + 2x^{11} + x^{12} + x^{13}\end{aligned}$$

So to answer the question

(a)
$$[x^9]\Phi_S(x) = 3$$

(b)
$$[x^{10}]\Phi_S(x) = 2$$

This is **stupid**. Note this is the same as

$$\Phi_S(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4 + x^6 + x^8)$$

Note that S is a composite product, that is

$$S = [x] = \left\{ \begin{array}{l} \text{L specifies how many loonies} \\ \text{T specifies how many toonies} \end{array} \right\}$$

so

$$\Phi_S(x) = \Phi_L(x)\Phi_T(x)$$

Let A be a set with weight function α. Let B be a set with weight function β. Let $S = A \times B$ and disjoint weight function $w : S \to \mathbb{N}$ where $w(a, b) = \alpha(a) + \beta(b)$. Then

$$\Phi_S(x) = \Phi_\alpha(x)\Phi_\beta(x)$$

Note the weight of a composite function must be the sum of the weights of the individual functions.

6 May 17, 2017

6.1 n Groups in Product Lemma

Example 6.1. You have 5 loonies and 4 toonies. I have 8 loonies and 3 toonies. How man ways can we make \$20 together?
Note that we cannot combine the loonies and toonies since WLOG X loonies from you and Y loonies from me is different than say Y loonies from you and X loonies from me.
Let S be the set of all combinations of loonies and toonies we can produce together.

Let L_1 be set of ways YOU can contribute loonies. T_1 YOU can contribute toonies. Similarly for L_2 and T_2 for ME. That is for example

$$L_1 = \{0, 1, 2, 3, 4, 5\} \text{ loonies}$$

For each of these sets, define the weight of a configuration as the dollar amount. So we have

$$S = L_1 \times T_1 \times L_2 \times T_2$$

or in other words, one must specify an element for all four subsets to specify an element of S.
We can use the **product lemma** here since it satisfies $w_S(a, b, c, d) = \sum_{i \in \{a,b,c,d\}} w_I(i)$, so

$$\begin{aligned}\Phi_S(x) &= \Phi_{L_1}(x)\Phi_{T_1}(x)\Phi_{L_2}(x)\Phi_{T_2}(x) \\ &= (x^0 + x^1 + x^2 + \ldots + x^5)(x^0 + x^2 + x^4 + \ldots + x^8)(x^0 + x^1 + x^2 + \ldots + x^8)(x^0 + x^2 + x^4 + x^6)\end{aligned}$$

We can use the finite geometric series formula

$$\Phi_S(x) = (\frac{x^0 - x^6}{1 - x})(\frac{x^0 - x^{10}}{1 - x^2})(\frac{x^0 - x^9}{1 - x})(\frac{x^0 - x^8}{1 - x^2})$$

So we can then solve for the coefficient

$$[x^{20}](\frac{x^0 - x^6}{1 - x})(\frac{x^0 - x^{10}}{1 - x^2})(\frac{x^0 - x^9}{1 - x})(\frac{x^0 - x^8}{1 - x^2})$$

as the number of ways to make \$20.

6.2 Compositions

Example 6.2. How many ways can you add up to 5 using 3 numbers ≥ 1?

(3, 1, 1)
(1, 3, 1)
(1, 1, 3)
(2, 2, 1)
(2, 1, 2)
(1, 2, 2)

Definition 6.1. A composition of n with k parts is a sequence $(c_1, \ldots, c_k)$ of positive integers $(1, 2, 3, \ldots)$ such that

$$c_1 + c_2 + \ldots + c_k = n$$

The numbers $c_1, \ldots, c_k$ are called the **parts** of the composition. k is the *number* of parts.
Let S be the weight of a composition $(c_1, \ldots, c_k)$ with exactly k parts. Define the weight of a composition in S as

$$w((c_1, \ldots, c_k)) = c_1 + \ldots + c_k$$

that is, the number of elements in S with weight n is the answer to our question.
Let $\mathbb{N}_{\geq 1} = \{1, 2, 3, \ldots\}$, or positive integers. Then

$$S = (\mathbb{N}_{\geq 1}) \times \ldots \times (\mathbb{N}_{\geq 1}) = (\mathbb{N}_{geq_1})^k$$

Define a weight function on $\mathbb{N}_{\geq 1}$

$$w : \mathbb{N}_{\geq 1} \to \mathbb{N}$$
$$\alpha(i) = i$$

Then we have

$$w((c_1, \ldots, c_k)) = \alpha(c_1) + \ldots + \alpha(c_k)$$

which we can use the product lemma.

$$\begin{aligned}
\Phi_S(x) &= (\Phi_{\mathbb{N}_{\geq 1}}(x))^k && \text{from Product Lemma} \\
&= (\sum_{i \geq 1} x^i)^k \\
&= (x(1-x)^{-1})^k && \text{from geometric series}
\end{aligned}$$

Now we must find $[x^n]$ from this last equation. Remember our negative binomial theorem

$$(1-x)^n = \sum_{k \geq 0} \binom{n+k-1}{k} x^k$$

So

$$\begin{aligned}
[x^n]x^k(1-x)^{-k} &= [x^{n-k}](1-x)^{-k} \\
&= \binom{(n-k)+k-1}{n-k} \\
&= \binom{(n-1}{n-k} \\
&= \binom{(n-1}{n-1-(n-k)} \\
&= \binom{n-1}{k-1}
\end{aligned}$$

therefore there are $\binom{n-1}{k-1}$ ways to compose n with k parts.

6.3 Isomorphic Solution to Compositions

Another way to find out compositions using an isomorphism does not involve generating functions.
Let there be an equation of the sum of n 1s with $n-1$ + signs. We must figure out a way to split this up into k groupings of sums.

$$(1 + 1 + 1 + \ldots + 1 + 1)$$

To group this into k groupings, we can choose $k-1$ of these $n-1$ + signs and convert them into commas (,).

$$(1 + 1, 1, 1 + \ldots + 1, 1 + 1)$$

Thus we have $\binom{n-1}{k-1}$ ways.

6.4 Compositions with Constraints

Example 6.3. Determine the number of compositions of n with k parts ($k \geq 1$) such that the first part is *odd* and the other parts are ≥ 2.
Solution Let us define

S set of compositions with k parts such that first part is odd and other parts ≥ 2

Weight function $w(c_1, \ldots, c_k) = c_1 + \ldots + c_k$

First part let $\mathbb{N}_{odd} = \{1, 3, 5, 7, \ldots\}$, set of positive integers

Other parts let $\mathbb{N}_{\geq 2} = \{2, 3, 4, \ldots\}$, set of positive integers that are ≥ 2

So we have for our set S

$$\begin{aligned} S &= \mathbb{N}_{odd} \times \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 2} \times \ldots \times \mathbb{N}_{\geq 2} \\ &= \mathbb{N}_{odd}(\mathbb{N}_{\geq 2})^{k-1} \end{aligned}$$

which translates to the weight functions

$$\begin{aligned} \Phi_{\mathbb{N}_{odd}}(x) &= x + x^3 + x^5 + x^7 + \ldots \\ &= \frac{x}{1 - x^2} \end{aligned}$$

and

$$\begin{aligned} \Phi_{\mathbb{N}_{\geq 2}}(x) &= x^2 + x^3 + x^4 + x^5 + \ldots \\ &= \frac{x^2}{1 - x} \end{aligned}$$

By Product Lemma we have the final weight function

$$\begin{aligned} \Phi_S(x) &= \Phi_{\mathbb{N}_{odd}}(x)\Phi_{\mathbb{N}_{\geq 2}}^{k-1} \\ &= (\frac{x}{1 - x^2})(\frac{x^2}{1 - x})^{k-1} \end{aligned}$$

So the answer to our problem is

$$[x^n]\Phi_S(x) = [x^n](\frac{x}{1 - x^2})(\frac{x^2}{1 - x})^{k-1}$$

6.5 Empty Parts

Example 6.4. How many compositions of n in which all parts are odd? **There could be 0 parts**.
By convention, there is one composition with 0 parts: () *and it is a composition of* 0.
Solution Let us define

S set of all configurations with only odd parts

Weight function $w(c_1, \ldots, c_k) = c_1 + \ldots + c_k$ for any $k \geq 0$

So the set of all compositions with k parts, all odd, is $(\mathbb{N}_{odd})^k$.

A composition in this S can have 0 parts *or* 1 part *or* 2 parts... That means

$$\begin{aligned} S &= (\mathbb{N}_{odd})^0 \cup (\mathbb{N}_{odd})^1 \cup (\mathbb{N}_{odd})^2 \cup \dots \\ &= \bigcup_{k \geq 0} (\mathbb{N}_{odd})^k \end{aligned}$$

By the Sum Lemma since these are disjoint sets

$$\begin{aligned} \Phi_S(x) &= \Phi_{(\mathbb{N}_{odd})^0}(x) + \Phi_{(\mathbb{N}_{odd})^1}(x) + \dots \\ &= \sum_{k \geq 0} \Phi_{(\mathbb{N}_{odd})^k}(x) \end{aligned}$$

By the product lemma for each $\mathbb{N}_{odd}$ set

$$\Phi_{(\mathbb{N}_{odd})^k}(x) = (\frac{x}{1-x^2})^k$$

So our final weight function for S is

$$\begin{aligned} \Phi_S(x) &= \sum_{k \geq 0} (\frac{x}{1-x^2})^k = 1 + (\frac{x}{1-x^2}) + (\frac{x}{1-x^2})^2 + \dots \\ &= \frac{1}{1 - \frac{x}{1-x^2}} \end{aligned}$$

Taking the coefficients as the answer

$$[x^n]\Phi_S(x) = [x^n]\frac{1}{1 - \frac{x}{1-x^2}}$$

7 May 23, 2017

7.1 10 Steps to Generating Series Problems

Example 7.1. How many compositions of n with k parts?

(0) Do you even need generating series to solve it?

(1) Identify parameters in problem (and any constants to be treated as parameter). In the example, n and k are the parameters.

(2) Create set of configurations S by removing one of the parameters. In the example, removing n means generating all configurations with k parts. Remove k means generating all configurations that compose to n. We'd remove n to generate k part configurations.

(3) Provide precise mathematical definition of S. Usually unions (Cartesian products) based on simpler sets $A_1, A_2, \dots$. In the example, we'd have k simpler sets and take the Cartesian product of them.

(4) Re-introduce the missing parameter as the weight function so that the problem can be stated as "How many elements of S with weight n?". In this example, we can re-introduce n as the missing parameter.

(5) Define weight functions on simpler sets $A_1, A_2, \dots$.

(6) Check that weight functions behave correctly for product lemma. In this example, they correspond to the composition of k parts to some n.

(7) Compute the generating series for each simpler set $\Phi_{A_1}(x), \Phi_{A_2}(x), \ldots$. For the example, this is the $\mathbb{N}_{\geq 1}$ set for each A_i.

(8) Use S description from step (3) using sum and/or product lemma to get an expression for $\Phi_S(x)$.

(9) Simplify $\Phi_S(x)$.

(10) Answer is $[x^n]\Phi_S(x)$.

7.2 Binary Strings

String composed of only 0s and 1s.

Example 7.2. How many binary strings are there of length n in which every 0 is followed by *exactly* 1, 2, or 3 1s?

Good "0110101110101"

Bad "101001"

Bad "1101111"

7.3 Concatenation of Strings

Concatenation of two strings $a, b \to ab$ is the string whose digits are the digits of a followed by the digits of b. For example, $a = 0110$ and $b = 11010$, then $ab = 011011010$ or $aa = 01100110$.

7.4 Concatenations of Sets of Strings

Let A, B be **sets of strings**. Then

$$AB = \{ab | a \in A, b \in B\}$$

For example, let $A = \{1, 01\}, B = \{1, 10\}$. Then

$$AB = \{11, 110, 011, 0110\}$$

Note however when there are duplicate elements from the Cartesian product (concatenation of $a \in A, b \in B$

$$\begin{aligned} BA &= \{11, 101, 101, 1001\} \\ &= \{11, 101, 1001\} \end{aligned}$$

Note that BA is **ambiguous** since we cannot deduce B and A now.

7.5 Empty String

The **empty string** ϵ is a string of length 0. That is

$$a\epsilon = a = \epsilon a$$

7.6 Blocks

$$0010100000001110 \to 00|1|0|1|0000000|111|0$$

Note we can form 7 *blocks* $00, 1, 0, 1, 0000000, 111, 0$ from this string.
Therefore, a **block** is a **maximal, non-empty substring** using only **one digit**. Maximal is defined as "as big as possible and can't be extended further". Therefore, 0000 in the block of 0000000 is not a block since it's not *maximal*.

7.7 Ambiguous vs Unambiguous

Concenation of sets of strings is very similar to the Cartesian product. Note

$$AB = \{ab|a \in A, b \in B\}$$

is very similar to

$$A \times B = \{(a, b)|a \in A, b \in B\}$$

where the difference is a comma.
If the map $f : A \times B \to AB$ defined by $f((a, b)) = ab$ is a bijection, then AB is **unambiguous**.

8 May 24, 2017

8.1 Union

Give A, B set of strings, $A \cup B$ performs as usual. Note $A \cup B$ is *unambiguous* if $A \cap B = \varnothing$.
Let $A = \{1, 01\}$ and $B = \{1, 10\}$. Then $AB \cup BA = \{11, 110, 011, 0110, 101, 1001\}$.

8.2 Unambiguity for Complex Expressions

Note a complicated expressions (made with $\cup$ and concatenation) we say the expression is *unamabiguous* iff each individual operation is unambiguous.
For example, $AB \cup BA$ is ambiguous since AB and BA are umabiguous. Furthermore, $AB \cap BA = \varnothing$ thus $AB \cup BA$ is unambiguous.

8.3 Empty string ϵ

Note ϵ is *not* included in any given set that does not include it explicitly (e.g. $\epsilon \notin \{0, 1\}$.

8.4 A^* ("Infinite" Unions)

Let A be a set of strings. Then A^* is

$$\begin{aligned} A^* &= \{\epsilon\} \cup A \cup AA \cup AAA \cup \dots \\ &= \bigcup_{k \geq 0} A^k \end{aligned}$$

where $A^0 = \{\epsilon\}$ and $A^k = AAA \dots A$ or the concatenation of k As (the kth concatenation power).
Note this can be ambiguous since A^k can be the cartesian product e.g. $A^k = A \times A \times \dots \times A$. In the context of strings, we usually mean **concatenation power**.
Aside: This * operator is very similar to the * operator in regex!

$\{0,1\}^*$ set of all binary strings (not $\{0,1\}^0$ includes or ϵ).

$\{0\}^*$ all strings with only 0s $\rightarrow \{\epsilon, 0, 00, 000, \ldots\}$

$\{1\}^*$ $\{\epsilon, 1, 11, 111, \ldots\}$

$\{0\}\{0\}^*$ $\{0, 00, 000, 0000, \ldots\}$ (all blocks of 0s)

$\{1\}\{1\}^*$ $\{1, 11, 111, 1111, \ldots\}$ (all blocks of 1s)

Note these are **all unambiguous** (with respect to * operation).
Take $\{0, 01, 10\}^*$. This is ambiguous since the element 010 can be either (0)(10) or (01)(0) which is unambiguous.

Example 8.1. From last class, we want a set of strings in which every 0 is followed by exactly 1,2, or 3 1s. We would like to formulate this in terms of an **unambiguous** A^*.
Note that $(\{0\}\{1, 11, 111\})^*$ fits our definition. However, note that we can start with any number of ones too. Therefore the correct answer is

$$\{1\}^*(\{0\}\{1, 11, 111\})^*$$

8.5 Brackets

() control order of operations

{} constructs a set

8.6 Generating Series for Sets of Strings

Unless otherwise specified, the weight function of a string is the **length of the string**.
There are three theorems

Sum Lemma If A, B are sets of strings and $A \cup B$ is unambiguous, then

$$\Phi_{A\cup B}(x) = \Phi_A(x) + \Phi_B(x)$$

Product Lemma (*Concetenation* vs. Cartesian products) If A, B are sets of strings and AB is unambiguous, then

$$\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$$

which is "weight preserving", that is

$$\Phi_{AB}(x) = \Phi_{A\times B}(x)$$

by the product lemma $\Phi_{A\times B}(x) = \Phi_A(x)\Phi_B(x)$.

9 May 26, 2017

9.1 Finite String Lemma

If A is a set of strings and A^* is unambiguous then

$$\Phi_{A^*}(x) = (1 - \Phi_A(x))^{-1}$$

Proof. Recall $A^* = \{\epsilon\} \cup A \cup AA \cup AAA \cup \ldots$. Then we have

A^* is unambiguous thus any two sets $X \cup Y$ must also be unambiguous (sum lemma)

$$\Phi_{A^*}(x) = \Phi_{\{\epsilon\}}(x) + \Phi_A(x) + \Phi_{AA}(x) + \ldots$$
$$= \sum_{k\geq 0} \Phi_{A^k}(x)$$

A^* is unambiguous thus A^k is unambiguous (product lemma)

$$= \sum_{k\geq 0} (\Phi_A(x))^k$$
$$= \frac{1}{1-\Phi_A(x)}$$

□

Example 9.1. Determine the number of binary strings of length n in which every 0 is followed exactly by 1,2 or 3 1s.
We saw that the set of binary strings of this type is

$$S = \{1\}^*(\{0\}\{1, 11, 111\})^*$$

This is unambiguous (talk about why later) thus

$$\begin{aligned}
\Phi_S(x) &= \Phi_{\{1\}^*}(x) \cdot \Phi_{(\{0\}\{1,11,111\})^*}(x) \\
&= \frac{1}{1-\Phi_{\{1\}}(x)} \cdot \frac{1}{1-\Phi_{(\{0\}\{1,11,111\})}(x)} \\
&= \frac{1}{1-x} \cdot \frac{1}{1-\Phi_{\{0\}}(x) \cdot \Phi_{\{1,11,111\}}(x)} \\
&= \frac{1}{1-x} \cdot \frac{1}{1-(x)(x+x^2+x^3)} \\
&= \frac{1}{(1-x)(1-x^2-x^3-x^4)}
\end{aligned}$$

Therefore our answer is

$$[x^n]\frac{1}{(1-x)(1-x^2-x^3-x^4)}$$

9.2 Standard Decomposition of Binary Strings

Recall that $\{0, 1\}^*$ is an unambiguous expression for the set of all binary strings.
You may also represent this in another way. For example, if we focus on 0 being "special" (**0-decomposition**), we may get

$$\{1\}^*(\{0\}\{1\}^*)^*$$

where the 0s are our fixed dividers. We may also have (for a 0-decomposition)

$$(\{1\}^*\{0\})^*\{1\}^*$$

Similarly for a **1-decomposition**

$$\{0\}^*(\{1\}\{0\}^*)^* \qquad \text{OR}$$
$$(\{0\}^*\{1\})^*\{0\}^*$$

Note these decompositions are *unambiguous.*
Can we use these decompositions to prove unambiguity for other unions? Yes!
Note in our previous example (0s followed by 1,2, or 3 1s), we replaced our $\{1\}*$ with a subset $\{1, 11, 111\}$.

$$\{1\}^*(\{0\}\{1\}^*)^*$$
$$\rightarrow \{1\}^*(\{0\}\{1, 11, 111\})^*$$

This is called a **restriction on the 0-decomposition**. This is *unambiguous* because it's a restriction (subset) of the unambiguous 0-decomposition.

9.3 Block Decomposition

Recall that blocks of 0s and 1s can be represented by $\{0\}\{0\}^*$ and $\{1\}\{1\}^*$, respectively (must have at least one element in each block).
Thus the block decomposition looks like

$$\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

Note the first and last $\{X\}^*$. This is because our block representation (inside the parentheses) enforces at least one 0 at the beginning and one 1 at the end; however, the string could start off with a block of 1s and/or end with a block of 0s. Thus we need to prepend/append the union set to cover all binary strings.
Another variation

$$\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$$

Example 9.2. Determin the number of binary strings of length n in which all blocks have odd length (are odd). Let S be the set of all binary strings in which all blocks are odd. That is

$$S = (\{\epsilon\} \cup \{1\}\{11\}^*)(\{0\}\{00\}^*\{1\}\{11\}^*)(\{\epsilon\} \cup \{0\}\{00\}^*)$$

Then we can use the **Finite Union Lemma** to find $\Phi_S(x)$.

10 May 29, 2017

10.1 Coefficient of Rational Functions

Note that for our previous enumeration problems, we have received answeres in the form of

$$[x^n]\frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials. This is called a *rational function.*

Remark: If $degree(f) \geq degree(g)$, then we can write

$$f(x) = q(x)g(x) + r(x)$$

for some unique:

(i) $q(x)$ is a polynomial

(ii) $r(x)$ is a polynomial such that $degree(r) < degree(g)$

Therefore for our initial answer

$$\begin{aligned}[x^n]\frac{f(x)}{g(x)} &= [x^n](q(x) + \frac{r(x)}{g(x)}) \\ &= [x^n]q(x) + [x^n]\frac{r(x)}{g(x)}\end{aligned}$$

Example 10.1. Find

$$[x^n]\frac{6x^4 - 5x^3 - 5x^2}{6x^2 - 5x + 1}$$

Long division gives us

$$\frac{6x^4 - 5x^3 - 5x^2}{6x^2 - 5x + 1} = (x^2 - 1) + \frac{1 - 5x}{1 - 5x + 6x^2}$$

For our remainder term, we can use partial fraction decomposition which gives us

$$\begin{aligned}(x^2 - 1) + \frac{1 - 5x}{1 - 5x + 6x^2} &= (x^2 - 1) + \frac{1 - 5x}{(1 - 2x)(1 - 3x)} \\ &= (x^2 - 1) + \frac{3}{1 - 2x} - \frac{2}{1 - 3x}\end{aligned}$$

Note our last 2 terms are simply $3(1 + 2x + 4x^2 + 8x^3 + \ldots)$ and $2(1 + 3x + 9x^2 + 27x^3 + \ldots)$. Thus we can formulate this generally for the coefficient x^n.

Generalization: We want $[x^n]\frac{f(x)}{g(x)}$ where $degree(f) < degree(g)$ and $g(x) = (1 - a_1x)(1 - a_2x) \ldots (1 - a_kx)$ where $a_1, \ldots, a_k$ are distinct. That is

$$\frac{f(x)}{g(x)} = \frac{c_1}{1 - a_1x} + \frac{c_2}{1 - a_2x} + \ldots + \frac{c_k}{1 - a_kx}$$

Note however that $(1 - a_ix)^{-1} = 1 + a_ix + a_i^2x^2 + \ldots$. Thus

$$[x^n]\frac{f(x)}{g(x)} = c_1a_1^n + c_2a_2^n + \ldots + c_ka_k^n$$

If $a_1, \ldots, a_k$ are not distinct, then remember our partial decomposition looks slightly different.

Example 10.2. Find

$$[x^n]\frac{2x^2 - 4x + 6}{x^3 - x^2 - x + 1}$$

thus

$$x^3 - x^2 - x + 1 = x(x^2 - 1) - 1(x^2 - 1) = (x - 1)^2(x + 1)$$

thus we have

$$\frac{2x^2 - 4x + 6}{x^3 - x^2 - x + 1} = \frac{3}{1 + x} + \frac{1}{1 - x} + \frac{2}{(1 - x)^2}$$

How do we deal with $(1-x)^{-2}$. Take the *negative binomial series.* That is

$$(1-x)^{-2} = \sum_{m\geq 0}\binom{m+2-1}{m}x^m = \sum_{m\geq 0}(m+1)x^m$$

expanding out our rational functions into geometric series

$$\frac{2x^2-4x+6}{x^3-x^2-x+1} = 3\sum_{i\geq 0}(-x)^i + \sum_{j\geq 0}x^j + 2\sum_{m\geq 0}(m+1)x^m$$

Taking the x^n coefficient

$$3(-1)^n + 1 + 2(n+1) = 3(-1)^n + 2n + 3$$

11 May 31, 2017

11.1 Recurrence Relations for Rational Functions

Note it's possible to decompose a rational function using *recurrence relations* (This is just a demonstration of how it works, later we will use a cleaner solution).

Example 11.1.

$$a_n = [x^n]\frac{6x^4-5x^3-3x^2}{6x^2-5x+1}$$

Let $A(x) = \sum_{n\geq 0} a_n x^n$. $A(x)$ is what we want to break our rational function down so we can write it in terms of infinite geometric series. So let's write

$$\begin{aligned} A(x) &= \frac{6x^4-5x^3-3x^2}{6x^2-5x+1} \\ \iff (6x^2-5x+1)A(x) &= 6x^4-5x^3-3x^2 \end{aligned}$$

Multiplying out $6x^2-5x+1$ with $A(x) = a_0 + a_1x + a_2x^2 + \ldots$, we want our final sum to be the LHS.

	a_0	a_1x	a_2x^2	a_3x^3	a_4x^4	a_5x^5
$6x^2$			$6a_0x^2$	$6a_1x^3$	$6a_2x^4$	$6a_3x^5$
$-5x$		$-5a_0x$	$-5a_1x^2$	$-5a_2x^3$	$-5a_3x^4$	$-5a_4x^5$
$+1$	a_0	a_1x	a_2x^2	a_3x^3	a_4x^4	a_5x^5
			$-3x^2$	$-5x^3$	$+6x^4$	

In order for this to be true, we have the following equations (along the column)

$$\begin{aligned} &a_0 = 0 \\ &-5a_0 + a_1 = 0 \\ &6a_0 - 5a_1 + a_2 = -3 \\ &6a_1 - 5a_2 + a_3 = -5 \\ &6a_2 - 5a_3 + a_4 = 6 \\ &6a_3 - 5a_4 + a_5 = 0 \\ &6a_4 - 5a_5 + a_6 = 0 \\ &6a_5 - 5a_6 + a_7 = 0 \end{aligned}$$

Note the last three terms are 0 that is $6a_{n-2} - 5a_{n-1} + a_n = 0$ for all $n \geq 5$.
So we have a *recurrence relation*

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \forall n \geq 5$$

with initial conditions

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= -5 \\ a_3 &= ? \\ a_4 &= ? \end{aligned}$$

(figure out ? values on your own)

Example 11.2. Let us go from a recurrence relation to the rational form.
Let $c_0 = 1, c_1 = 0, c_2 = 0$ and $c_n = 7c_{n-1} - 16c_{n-2} + 12c_{n-3}$ for all $n \geq 3$.
Find a rational function $\frac{p(x)}{q(x)}$ such that $c_n = [x^n]\frac{p(x)}{q(x)}$.
Solution Let $C(x) = \sum_{n\geq 0} c_n x^n$. Multiplying out the given recurrence equation (setting it to 0) with $C(x)$.

$C(x) =$	c_0	c_1x	c_2x^2	c_3x^3	c_4x^4	c_5x^5
$-7xC(x) =$		$-7c_0x$	$-7c_1x^2$	$-7c_2x^3$	$-7c_3x^4$	$-7c_4x^5$
$+16x^2C(x) =$			$+16c_0x^2$	$+16c_1x^3$	$+16c_2x^4$	$+16c_3x^5$
$-12x^3C(x) =$				$-12c_0x^3$	$-12c_1x^4$	$-12c_2x^5$
$(1-7x+16x^2-12x^3)C(x) =$	c_0	$(c_1-7c_0)x$	$+(c_2-7c_1+16c_0)x^2$	$+0x^3$	$+0x^4$	$+0x^5$
$=$	1	$-7x$	$+18x^2$			

Thus we have

$$C(x) = \frac{1-7x+18x^2}{1-7x+16x^2-12x^3}$$

Factoring the denominator

$$1-7x+16x^2-12x^3 = (1-2x)^2(1-3x)$$

So then by partial decomposition we get

$$\frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-3x}$$

which maps to the coefficient

$$c_n = (\alpha + \beta n)2^n + \gamma 3^n$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and for ALL $n \geq 0$.
This is called the **general solution** to the recurrence (note we only use the denominator which is independent of our initial conditions).

$$c_n = 7c_{n-1} + 16c_{n-2} + 12c_{n-3}$$

This means that every sequence that satisfies this recurrence is of this form and every sequence of this form satisfies the recurrence.

Note that the original problem had initial conditions $c_0 = 1, c_1 = 0, c_2 = 2$. Compare this with $c_n = (\alpha+\beta n)2^n + \gamma 3^n$

$$n = 0 : 1 = (\alpha + \beta \cdot 0)2^0 + \gamma 3^0$$
$$n = 1 : 0 = (\alpha + \beta \cdot 1)2^1 + \gamma 3^1$$
$$n = 2 : 2 = (\alpha + \beta \cdot 2)2^2 + \gamma 3^2$$

Three equations three unknowns, solve for α, β, γ.

12 June 2, 2017

12.1 Notes on Recurrence Relations

How many initial conditions do we need?

Theorem 12.1. Suppose

$$\sum_{i \geq 0} a_i x^i = \frac{p(x)}{q(x)}$$

is a rational function and

$$q(x) = 1 + \sum_{j=1}^{k} q_j x^j$$

Then $\{a_n\}$ satisfies the recurrence

$$a_n = -\sum_{j=1}^{k} q_j a_{n-j}$$

for $n \geq max(deg(q(x), 1 + p(x)))$. The initial conditions are determined by $p(x)$. (You can see this holds for our previous examples).

12.2 Characteristic Polynomial

For the recurrence

$$a_n + \sum_{j=1}^{k} q_j a_{n-j} = 0$$

the **characteristic polynomial** is

$$x^k + \sum_{j=1}^{k} q_j x^{k-j} = 0$$

12.3 Presenting Recurrence Solutions

Example 12.1. Solve the recurrence

$$c_n = 7c_{n-1} - 16c_{n-2} + 12c_{n-3}$$

with initial conditions $c_0 = 1, c_1 = 0, c_2 = 2$.

Solution The characteristic polynomial is

$$x^3 - 7x^2 + 16x - 12 = (x-2)^2(x-3)$$

So the roots are 2, 2, 3. Note that the characteristic polynomial does not map 1-1 to our derivation with the table thing, however, it works out in the end (think of factoring the denominator then taking the partial fraction decomposition, which translates to a bunch of geometric series r_i^n where r_i is a given root. Note this always works since there'll always be k roots by the fundamental theorem of algebra).
That is our general solution would be

$$c_n = (\alpha + \beta n)2^n + \gamma 3^n$$

Let's plug in our initial conditions, that is $n = 0, 1, 2$

$$\begin{aligned} 1 = c_0 &= (\alpha + \beta \cdot 0)2^0 + \gamma 3^0 \\ 0 = c_1 &= (\alpha + \beta \cdot 1)2^1 + \gamma 3^1 \\ 2 = c_2 &= (\alpha + \beta \cdot 2)2^2 + \gamma 3^2 \end{aligned}$$

Example 12.2. For a polynomial with four 2 roots, three 1 roots, and two 7 roots

$$(\alpha + \beta n + \gamma n^2 + \partial n^3)2^n + (\epsilon + \phi n + \mu n^2)1^n + (\theta + \sigma n)7^n$$

12.4 Crazy Dice Problem

Find 2 six-sided dice (not ordinary, can be different) that gives the same probability table as two ordinary dice.

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	...							

Solution Let S be the set of sides of an ordinary die $(1, 2, 3, 4, 5, 6)$.
Let A be the set of sides on the first crazy dice. Similarly B for the second.
In each case, define weight of a side to be the number on it. Thus we have

$$\Phi_S(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$$

Goal is to find $\Phi_A(x)$ and $\Phi_B(x)$.
Key Point Total number of ways to roll n on crazy dice should be *equal* to total # of ways to roll on 2 ordinary dice. Translating this problem into generating series (where weight is sum of sides)

$$\begin{aligned} \Phi_{A\times B}(x) &= \Phi_{S\times S}(x) \\ \Phi_A(x)\Phi_B(x) &= \Phi_S(x)^2 \end{aligned}$$

by product lemma.
Converting $\Phi_S(x)^2$ into its geometric series rational form we get

$$\therefore \Phi_A(x)\Phi_B(x) = (\frac{x - x^7}{1-x})^2$$

To find $\Phi_A(x)$ and $\Phi_B(x)$, we can factor the RHS and distribute the factors in a way such that we get a reasonable answer for each.

Factoring the RHS

$$\begin{aligned}
\frac{x-x^7}{1-x} &= \frac{x(1-x^6)}{1-x} && \text{factor x} \\
&= \frac{x(1-x^3)(1+x^3)}{1-x} && \text{difference of squares} \\
&= \frac{x(1-x)(1+x+x^2)(1+x)(1-x+x^2)}{1-x} && \text{difference/sum of cubes} \\
&= x(1+x+x^2)(1+x)(1-x+x^2) && \text{cancelling denom}
\end{aligned}$$

Since we square the RHS, we get

$$\therefore \Phi_A(x)\Phi_B(x) = x \cdot x(1+x)(1+x)(1+x+x^2)(1+x+x^2)(1-x+x^2)(1-x+x^2)$$

Note that the # on each die must be ≥ 1. Also $\Phi_A(1) = \Phi_B(1) = 6$ (Since they must have 6 sides each). Furthermore $\Phi_A(0) = \Phi_B(0) = 0$ (there can't be any constants).
Thus the only case where this holds is

$$\begin{aligned}
\Phi_A(x) &= x(1+x)(1+x+x^2)(1-x+x^2)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8 \\
\Phi_B(x) &= x(1+x)(1+x+x^2) = x + 2x^2 + 2x^3 + x^4
\end{aligned}$$

So we have our crazy dice with sides 1,3,4,5,6,8 and 1,2,2,3,3,4. We can extend this for multiple dice with different number of sides.

13 June 5, 2017

13.1 Definition of Graphs

A graph G consists of a finite set $V(G)$ called the *vertices* and another finite set $E(G)$ called the *edges*.
An edge is an unordered pair of distinct vertices.

Example 13.1. Consider $V(G) = \{a, b, c, d, e\}$. Then perhaps $E(G) = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, e\}\}$.
This can be drawn in multiple ways (but still satisfies the set of edges and vertices).

Note that $V(G)$ is a finite set (no infinite graphs). $E(G)$ is a *true* set (no concept of multiple edges for two given vertices). Furthermore, an edge consists of *distinct vertices* (no loops). Edges are also unordered pairs of vertices (undirected).

13.2 Edge Terminologies

If $e = \{u, v\} \in E(G)$ is an edge, then

- u and v are *adjacent*
- u and v are *neighbours*
- e is *incident* with u (and with v)
- e *joins* u and v

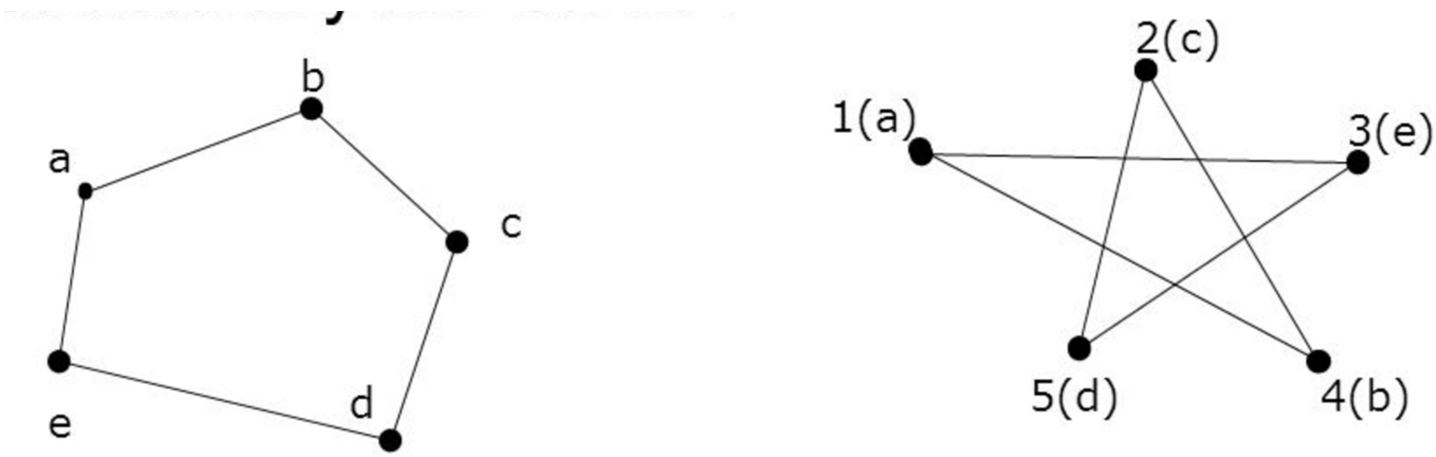
13.3 Neighbours and Degree of Vertex u

The set of all neighbours of a vertex $u \in V(G)$ is denoted $N(u)$.
The **degree** of $u \in V(G)$ is $deg(u) = |N(u)|$ (also the number of edges incident with u). In the previous example, $N(a) = \{b, d, e\}$ or $deg(a) = 3$.

13.4 Graph Morphisms

N graphs be not be the same but have the same structure.

Figure 13.1: Isomorphic Graphs



In **Figure ??**, all vertices have degree 2 and each vertex from each graph can be mapped one-to-one to another vertex in the other graph. These graphs are **isomorphic**.

Definition 13.1. Two graphs G_1, G_2 are isomorphic if there exists a **bijection**

$$f : V(G_1) \to V(G_2)$$

that "preserves adjacencies". That is

$$\{u, v\} \in E(G_1) \iff \{f(u), f(v)\} \in E(G_2)$$

The map f is called an **isomorphism**.

In the previous example, we have $f : V(G_1) \to V(G_2)$ such that

$$\begin{aligned} f(a) &= 1 \\ f(b) &= 4 \\ f(c) &= 2 \\ f(d) &= 5 \\ f(e) &= 3 \end{aligned}$$

Then you would check each edge in G_1 incident to their respective two vertices holds in G_2.

14 June 7, 2017

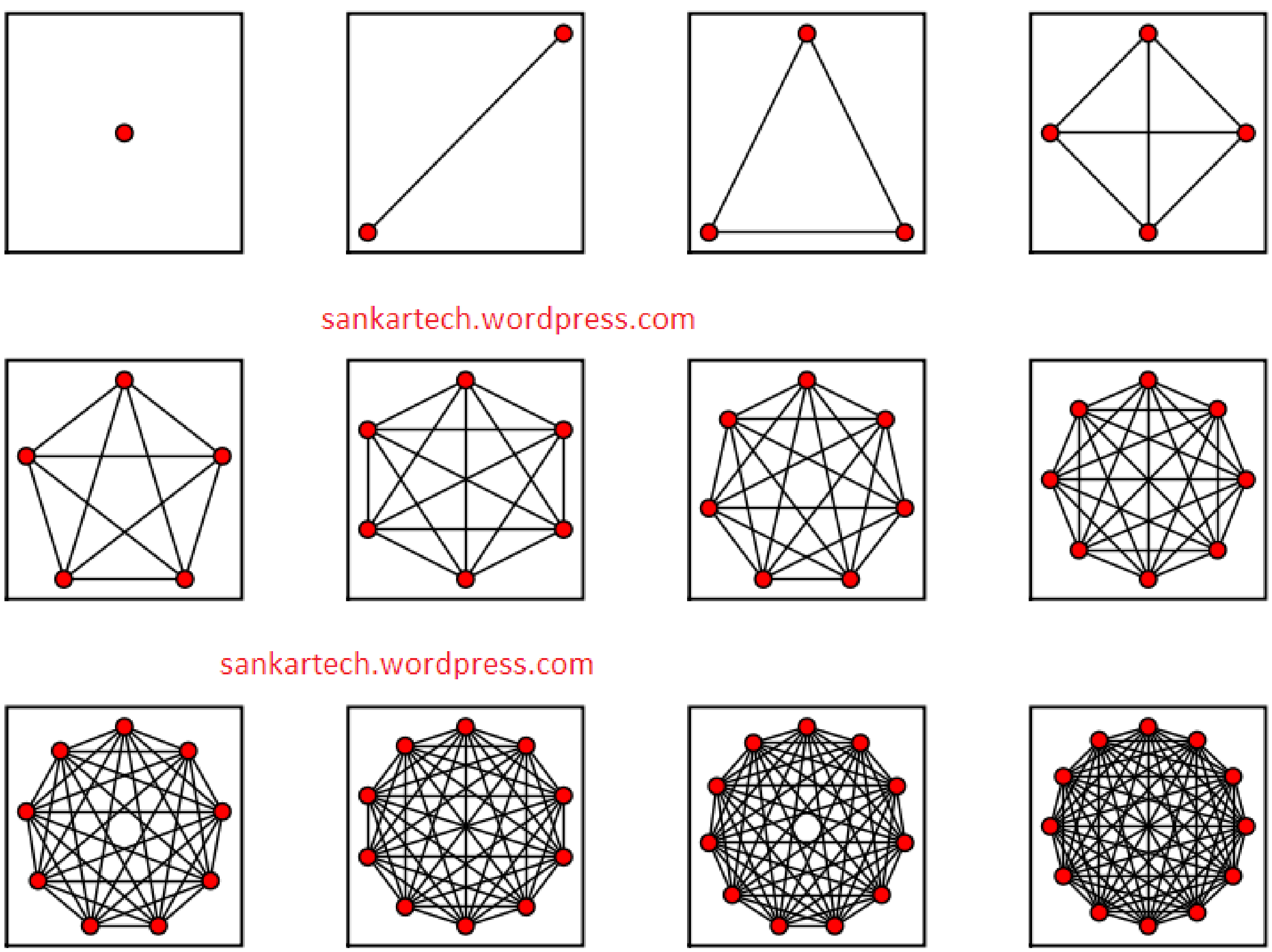
Graphs must have labels!

14.1 Special Graphs

K-regular graph A graph is *k-regular* if every vertex has degree k.

Complete (K-regular) graph A graph where every pair of distinct vertices are adjacent. A complete graph K_p with p vertices is also a K-regular graph ($p-1$ regular). Thus there are $\binom{p}{2}$ edges.

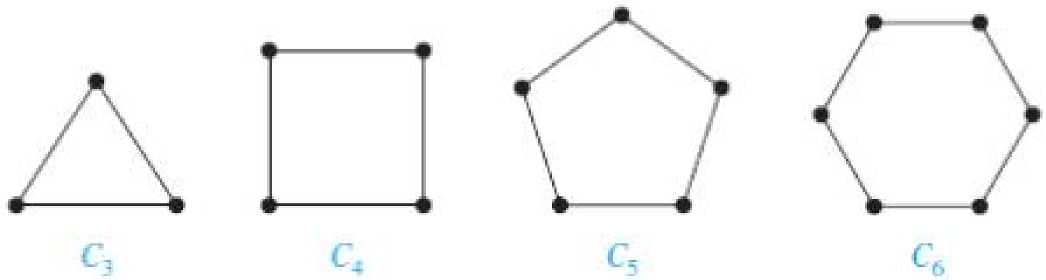
Figure 14.1: Complete K_p graphs with p vertices



Cycle (2-regular) Formal description later.

Petersen graph

Figure 14.2: Cycle graphs C_p with p vertices



Bipartite graph A graph G is bipartite if $V(G)$ can be partitioned into (A, B) such that every edge of G joins a vertex in A to a vertex in B (a given vertex may be incident to multiple edges). Note that the pair (A, B) is a bipartition.

Complete bipartite graph A graph $K_{m,n}$ is a complete bipartite graph if there is a bipartition (A, B) with $|A| = m, |B| = n$ and *every* vertex in A is adjacent to *every* vertex in B. Thus $|V(K_{m,n})| = m + n$ and $|E(K_{m,n})| = mn$.

N-cube graph A graph Q_n is n-cube if:

Let $V(Q_n)$ be the set of all binary strings of length n. Edges: two vertices (strings) are adjacent if they differ in exactly one position. For example, $\{011011, 011001\} \in E(Q_6)$. Note that $V(Q_n) = 2^n$. Furthermore, note that n-cube graphs are also *regular* where $k = n$ or Q_n is n-regular. Furthermore, note that Q_n is bipartite where $A =$ strings with even # of 1s, $B =$ strings with odd # of 1s.

Planar graph A graph is planar if it can be drawn with *no edges crossing*. Some graphs (e.g. $K_4, Q_3, K_{2,3}$) can be drawn planar. A lot of graphs cannot however. We will prove this later.

14.2 Handshaking Lemma

If G is a graph with q edges then

$$\sum_{v \in V(G)} deg(v) = 2q$$

Proof. Two ways of counting the # of half-edges in G: formally, count pairs (v, e) such that e is incident with v.

Example 14.1. How many edges in Q_n?
There are 2^n vertices and each vertex has degree n. Therefore $2q = \sum_{v \in V(Q_n)} deg(v) = n2^n$, so $q = n2^{n-1}$.

□

15 June 9, 2017

15.1 Teams and Games Graph Problem

There are 11 teams a 60 hours of time allotted. Each game takes 2 hours. How many games can each team play if all teams must play the same # of games?
3,4,5, or 6 games?

We can solve this using graphs.
Let the vertices be teams and the edges be games played between two teams. Note that the graph must be k-regular (where degree of each vertex is k.
How many edges are there in the graph? There will be $q = \frac{11k}{2}$ edges or in other words $2q = \sum_{v \in V(G)} deg(v)$. Note that q is not an integer if $k = 5$. $k = 6$ is too large obviously. Thus the answer is $k = 4$.

15.2 Number of Odd Degree Vertices

In every graph, the number of vertices of odd degree is even.

Proof. Otherwise $\sum_{v \in V(G)} deg(v)$ is odd (odd number of odd degrees = odd even with the even degrees) which is impossible (since each edge adds two degrees to the sum, can't have half an edge). □

15.3 Walks

Definition 15.1. Let x, y be two vertices ($x = y$ is allowed) in a graph G. A **walk** in G from x to y is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ where $v_0 = x, v_n = y$ and $e_i = \{v_{i-1}, v_i\}$ for all $i = 1, \ldots, n$.
Note: sometimes we omit the edges when we specify a walk and simply write $v_0, v_1, \ldots, v_n$. Note this would not work if there are multiple edges (which is not permitted in this course).
Note: given a vertex a, the sequence of vertices a, a is not a walk nor path since they are not different vertices (also no one edge loops back to a itself).

Definition 15.2. The **length** of the above walk is n. Note it is the number of edges, NOT vertices.

Definition 15.3. A walk $v_0, v_1, \ldots, v_n$ is **closed** if $v_0 = v_n$.

15.4 Paths

Definition 15.4. A **path** is a walk with no repeated vertices.

15.5 Walk → Path

Theorem 15.1. Let G be a graph and let $x, y \in V(G)$. If there is a walk from x to y, then there is a **path** from x to y.

Proof. Assume there exists at least one walk from x to y.
Consider a shortest walk $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$. I claim this must be a path.
Suppose it is not a path. Then there exists $i < j$ such that $v_i = v_j$. But then $v_0, e_1, v_1, e_2, v_2, \ldots, |v_i \ldots|, v_j, e_{j+1}, v_{j+1}, e_{j+2}, v_j$ is a walk with length $< n$ (once we take out the block).
Hence this is not the shortest walk and thus it is a contradiction. □

16 June 12, 2017

16.1 Transitivity (and Equivalence) of Paths

Theorem 16.1. Suppose $x, y \in V(G)$. If there is a *path* from x to y and a *path* from y to z, then there is a *path* from x to z.

Proof. If $v_0, v_1, \ldots, v_n$ is a path from x to y and $u_0, u_1, \ldots, u_m$ is a path from y to z, then $v_0, v_1, \ldots, v_n, u_1, \ldots, u_m$ is a *walk* from x to z.
By the previous theorem, there is a path from x to z. □

Remark: The relation $x \equiv y$ if there is a walk from x to y is an *equivalence relation.* In other words (in terms of the three properties of equivalence relations: reflexitivity, symmetry, and transitivity)

1. For any vertex x there is a walk from x to x (assuming there is more than 1 vertex)
2. If there is a walk from x to y, then there is a walk from y to x
3. If there is walk from x to y and from y to z, then there is a walk from x to z

Note paths only satisfy transitivity and symmetry (you can't have a path from a vertex to itself).

16.2 Connected Graphs

Definition 16.1. A graph is **connected** if *for any two vertices* $x, y \in V(G)$ there is a path from x to y.

To prove a graph is connected use the **"Hub model"**.

Theorem 16.2. Suppose there exists a vertex $x \in V(G)$ (the hub) such that for every vertex y, there is a path from x to y. Then G is connected.

Example 16.1. The n-cube graph Q_n is connected. Recall vertices are connected if they differ by 1 position in their binary string.

Proof. Let $x = 0000\ldots0$. Pick any vertex $y \in V(Q_n)$. Let $i_1, i_2, \ldots, i_k$ be the positions of 1s in y. Let x_j be the string with 1s in positions $i_1, i_2, \ldots, i_j$ and 0s elsewhere for $j = 0, \ldots, k$.
Thus $x_0 = x$ and $x_k = y$ and $x_0, x_1, x_2, \ldots, x_k$ is a path from x to y. Therefore Q_n is connected by "Hub Theorem". □

How do we prove a graph is **not connected**? We will show this later.

16.3 Subgraph

Let G be a graph. A subgraph H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Note as per our definition of graphs, H need not be connected (that is there can be singular vertices without any incident edges). Note the vertices and edges in any *walk* gives a subgraph. If the walk is a *path*, that subgraph is isomorphic to somehing like **Figure 16.1** where two vertices have degree 1 and the others have degree 2. A path graph is *connected.*

16.4 Spanning (Subgraphs)

Let G be a graph and H be a subgraph. We say H is **spanning** if $V(G) = V(H)$ (**note** $V(H)$ **need not be connected!! What we think of "spanning" as in Prim's MST is a *spanning tree***).

16.5 N-Cycle

Definition 16.2. An **n-cycle** is a graph C_n such that

$$V(C_n) = \{v_1, \ldots, v_n\}$$
$$E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_1, v_n\}\}$$

For example, a 6-cycle graph has 6 vertices connected in a cycle.

16.6 Cycle in Graph

A **cycle** in a graph G is a subgraph of G that is a cycle.
The **girth** $g(G)$ of a graph G is the length of the shortest cycle. If there are no cycles, then $g(G) \to \infty$.
Note a *spanning cycle* in a graph is called a **Hamilton Cycle**. It is easy to verify a Hamilton cycle but may be difficult to find or certify there is not a Hamilton cycle.

16.7 Cycle Walks

Note in any cycle C_n, there are $2n$ closed walks around the circle (For each of the n vertices, you can walk one way or the other way). These are called **cycle walks**.

16.8 Component of a Graph

A **component** of a graph is a maximal (i.e. cannot be extended, NOT maximum) connected subgraph.
Note a graph is connected **if and only if it has one component**.

17 June 14, 2017

17.1 Cut Induced by X

Given a partition (X, Y) of graph G such that there are no edges having an end in X and an end in Y, then there is no path from any vertex X to any vertex Y. If X and Y are both non-empty, then G is *not connected*.

Definition 17.1. Given a subset X of $V(G)$, a **cut** $\delta(X)$ induced by X is the set of edges that have exactly one end in X (In other words, the set of edges coming out of X, and possibly into Y in a partition (X, Y) of G).

17.2 Connected Graph via Proper Non-Empty Subset

Theorem 17.1. A graph G is connected $\iff$ for every *proper non-empty subset* $X \subset V(G)$ we have $\delta(X) \neq \varnothing$.

Note **proper** means $X \neq V(G)$ and *non-empty* means $X \neq \varnothing$. Together these should be at least one vertex in X and at least one vertex *not* in X.
See course notes for **details**.
Note: this gives us a strategy for proving a graph is *not connected*, that is find a set $X \subset V(G)$ such that $X \neq \varnothing$, $X \neq V(G)$, and $\delta(X) = \varnothing$ (the cut of X has no outwards edges).

17.3 Eulerian Circuit/Tour

Definition 17.2. An **Eulerian** circuit of a graph is a *closed walk that contains every edge of G once.*

Theorem 17.2. Let G be a connected graph. G has a Eulerian circuit if and only if each vertex has *even degree.*

Proof. Proof by induction. See course notes for details. □

Note that a graph with no edges and only vertices still has an Eulerian Tour! (a walk from vertex a to itself visits "every edge")

17.4 Bridges

Definition 17.3. Let G be a graph, $e \in E(G)$. Define $G - e$ to be the *spanning* subgraph of G ($V(G-e) = V(G)$) such that $E(G-e) = E(G) \setminus \{e\}$.
A **bridge** in a graph G is an edge $e \in E(G)$ such that $G - e$ has more components than G.

17.5 Deletion of Bridges → 2 Components

Lemma 17.1. Let G be a connected graph (1 component). If $e \in E(G)$ is a bridge $\iff$ $G - e$ has **2 components exactly**.
If $e = \{x, y\}$ then x and y are in different components of $G - e$.
Note: This means for non-connected graphs, deletion of a bridge increases the number of components by 1.

Proof. Let $z \in V(G)$. We will show that either z is in the same component as x in $G - e$ or z is in the same component as y in $G - e$.
Since G is connected, there is a path from z to x ($v_0, e_1, v_1, \ldots, e_n, v_n$ where $v_0 = z, v_n = x$).
If this path does not include e, then it is also a path in $G - e$. In this case z is in the same component as x in $G - e$.
If this path includes e, since $v_n = x, e = e_n, v_{n-1} = y$. Thus $v_0, e_1, v_1, \ldots, e_{n-1}, v_{n-1}$ this is a path from z to y in $G - e$. In this case z is in the same component as y in $G - e$.
Therefore $G - e$ has at most 2 components.
But e is a bridge so $G - e$ has at least 2 components.
Therefore $G - e$ has *exactly* 2 components: the component of x and the component of y. □

Theorem 17.3. Let G be a graph. An edge $e \in E(G)$ is a bridge $\iff$ e is not contained in any cycle.

Proof. Forward direction:
Assume e is a bridge. Suppose e is contained in a cycle. Write $e = \{x, y\}$. Let $x, e, y, e_2, v_2, \ldots, x$ be the cycle walk. Then $y, e_2, v_2, \ldots, x$ is a path from x to y in $G - e$.
Therefore x and y are in the same component of $G - e$. This means e **is not a bridge** by the previous lemma which is a contradiction.
Backwards direction:
Assume e is not in a cycle. Suppose e is not a bridge. Then x and y are in the same component of $G - e$. Therefore there is a path $y, e_2, v_2, \ldots, x$ from y to x in $G - e$. These vertices and edges together with e make a cycle, which is contradiction. □

Theorem 17.4. If $x, y \in V(G)$ and there exists two distinct paths from x to y, then G contains a cycle.
See course notes for details.

18 June 19, 2017

18.1 Certifying Properties

To show a graph contains a property, we can for example:

Connectedness Show a path exists between every pair of distinct vertices.

Non-connectedness Product a cut (two non-empty sets of vertices A and B that partition $V(G)$ such that no edge of G joins a vertex in A to a vertex in B

Bridge Show that $e = uv$ is a bridge by producing cut (A, B) where $u \in A$ and $v \in B$

Non-bridge e is not a bridge if a cycle contains e

18.2 Trees

Definition 18.1. A **tree** is a connected graph with no cycles.

Lemma 18.1. There is a unique path between every pair of vertices u and v in a tree T.

Proof. For any 2 vertices u and v in T, there is at least 1 path joining them since T is connected. Since T has no cycles, there is at most one path by previous corollary (two distinct paths $\rightarrow$ cycle). □

Lemma 18.2. Every edge of a tree T is a *bridge*.

Proof. An edge e of T is not in a cycle, so by previous theorem (bridge $\iff$ not in a cycle) e is a bridge. □

18.3 Forest

Definition 18.2. A graph with no cycles (which may not be connected) is called a **forest**, where every component is a *tree*.

18.4 Trees with At Least 2 Vertices of Degree One

Theorem 18.1. A tree with at least 2 vertices has at least two vertices of *degree one*.

Proof. Find and construct the longest path $w_0, w_1, \ldots, w_n$ in tree T between vertices $v = w_0$ and $v = w_n$. This path is at least length one since any edge in a non-trivial tree is a path of length one, so $u \neq v$.
There is one vertex adjacent to v in the path, that is w_{n-1}. Suppose $deg(v) > 1$, then there is another vertex w adjacent to v. Vertex w cannot be in the path since this would imply a cycle, thus we can extend the "longest path" by adding edge $\{v, w\}$ to it. This is a contradiction of the longest path thus $deg(v) = 1$. Similarly $deg(u) = 1$. □

18.5 Number of Edges and Vertices in Trees

Theorem 18.2. If T is a tree, then $|E(T)| = |V(T)| - 1$.

Proof. Proof by mathematical induction on p the number of vertices.
During the proof for p vertices (assuming IH holds for fewer than p vertices), note by previous theorem there is vertex u in T with degree one. Let v be the neighbour of u in T and let e be the edge $\{u, v\}$ (e is a bridge). Thus $T \setminus e$ is not connected and thus it has exactly 2 components, one of which is the vertex u (since it had degree one). The other graph has $p - 1$ vertices, is connected with no cycles and is thus a tree. By the IH, it has $p - 2$ edges, thus $|E(T)| = |V(T)| - 1$. □

Note the converse is not true! For example, if you have 4 vertices and 3 edges, you can have a triangle and a vertex all by itself.
If however the graph is connected and has $p-1$ edges, then it is a tree.

18.6 Number of Vertices of Degree One $\geq$ Max Degree

Theorem 18.3. Let n_r denote the number of vertices of degree r in T. Note that if T tree contains a vertex of degree r, then $n_1 \geq r$.

Proof. Note from a previous corollary where we note $\sum_{v\in V(G)} deg(v) = |E(G)|$. Let $p = |V(T)|$ and assume $p \geq 2$. From the above theorem, we have

$$2p-2 = \sum_{v\in V(T)} deg(v)$$

therefore

$$-2 = \sum_{v\in V(T)} deg(v) - 2 = \sum_{r=0}^{n-1} n_r(r-2)$$

Note $n_0 = 0$ since a tree is connected, so we get

$$-2 = -n_1 + \sum_{r\geq 3}(r-2)n_r$$
$$n_1 = 2 + \sum_{r\geq 3}(r-2)n_r$$

Since $(r-2)n_r \geq 0$ when $r \geq 3$, it follows $n_1 \geq 2$. Furhtermore if T contains a vertex of degree r, then $n_1 \geq 2 + (r-2) = r$. □

18.7 Leaf

Definition 18.3. A vertex of degree 1 in a tree is called a **leaf**.

19 June 21, 2017

19.1 Spanning Trees

Definition 19.1. A spanning subgraph which is also a tree is called a **spanning tree**.

Theorem 19.1. A graph G is connected if and only if it has a spanning tree.

Proof. Reverse direction:
We are given the graph has a spanning tree T. By a previous Lemma (unique path between every pair of vertices in tree T), each of these paths is also contained in G, and G has the same vertices as T (spanning), so G is connected.
Forward direction:
We are given G is connected. If G has no cycles, then G itself is a spanning tree of G. Otherwise G has a cycle. Remove any edge e of some cycle. Then $G-e$ is still connected (a bridge cannot be in a cycle by previous theorem). Note $G-e$ has fewer cycles than G. Repeat this process until we have a connected, spanning subgraph with no cycles. This subgraph is a spanning tree of G. □

Corollary 19.1. If G is connected with p vertices and $q = p - 1$ edges, then G is a tree.
Note from previous theorem, if G is connected then it contains a spanning tree T. Note T has p vertices thus by previous theorem it has $p - 1$ edges. Note G only has $p - 1$ edges, thus $G = T$, so G is a tree.

19.2 Creating Cycles From Spanning Tree

Theorem 19.2. If T is a spanning tree of G and e is not an edge in T, then $T + e$ contains exactly one cycle C. Moreover, if e' is any edge on C, then $T + e - e'$ is also a spanning tree of G.

Proof. Let $e = \{u, v\}$. Any cycle in $T + e$ must use e since T has no cycles. Such a cycle consists of e along with a $u, v - path$ in T. By previous lemma (unique path for pairs of vertices in tree), a unique path for $u, v - path$ in T, hence there is exactly one cycle C in $T + e$.
If e' is any edge in C, then e' is not a bridge. So $T + e - e'$ is still connected. Since it has $n - 1$ edges, by previous corollary it is a tree. □

19.3 Disconnecting and Connecting Spanning Tree

Theorem 19.3. If a T is a spanning tree of G and e is an edge in T, then $T - e$ has 2 components. If e' is in the cut induced by one of the components, then $T - e + e'$ is also a spanning tree of G.

Proof. The first is a consequence of the lemma where every edge is a bridge in a tree.
Let C_1, C_2 be the two components of $T - e$. Suppose $e' = \{u, v\}$ where $u \in V(C_1)$ and $u \in V(C_2)$.
Let $x \in V(C_1)$. For any $y \in V(C_1)$ there exists $x, y - path$ since C_1 is connected.
If $y \in V(C_2)$, since C_1, C_2 are each connected, there exists an $x, u - path$ P_1 and a $v, y - path$ P_2. Then P_1, e, P_2 form an $x, y - path$, thus $T - e + e'$ is connected. Since $T - e + e'$ has $n - 1$ edges, it is a tree. □

19.4 Bipartite Graphs and Odd Cycles

Theorem 19.4. A graph G is bipartite if and only if G has *no odd cycles*.

Proof. **Forward direction:**
If G is bipartite, then every subgraph of G is bipartite, including all cycles. For a cycle to be bipartite it must be even.
Backwards direction:
Assume G is connected. Let T be a BFST of G rooted at r. We will prove

1. If G has an edge $e = \{x, y\}$ (in the original graph) joining two vertices of the same level ($level(x) = level(y)$), then it has an odd cycle

2. If G has no edge joining two vertices of the same level, then G is bipartite

Putting these two together gives us our result.

1. There are two paths from x to r: $x, pr(x), \ldots, r$ and $x, y, pr(y), \ldots, r$. Within these is a cycle (it must be odd since they are both paths of equal length (x and y are on the same level), then it must be odd.

2. If G has no edge joining 2 vertices of the same level, then for every edge $\{x, y\} \in E(G)$ we have $|level(x) - level(y)| = 1$ (note if we have an edge that jumps multiple levels, then that would be in the BFST instead so this holds). Exactly one of x, y has odd level (non-empty sets). So we take $A = \{v \in V(G) | level(v) odd\}$ and $B = \{v \in V(G) | level(v) even\}$, thus (A, B) is a bipartition.

□

20 June 23, 2017

20.1 Breadth-First Search Trees

The algorithm is as follows with input graph G and a vertex (the "root" of the BFST)
We loop on the following

- Take next vertex in queue: x
- Add any neighbors of x that aren't already in the tree to the tree (BFST)
- Add the new vertices into the queue

Note: when adding vertices to the tree and queue, the algorithm does not say anything about the order. For *homework/tests*, one must follow the prescribed order when there is a choice.

Definition 20.1. A *BFST* is the output of the BFST algorithm.

Theorem 20.1. If G is a connected graph, then the BFST algorithm will produce a spanning tree. (If G is not connected, then there is no spanning tree so it doesn't produce one).
Thus this is an if and only if statement.

Proof. **Forward directions:**
Assume G is connected. Let T be a BFST. Consider $\delta(V(T))$ (cut induced by vertices of T).
In the BFST algorithm, we ensure that for every vertex $x \in V(T)$ all neighbors of x are in $V(T)$. So $\delta(V(T)) = \varnothing$.
Since G is connected, this is only possible if $V(T) = \varnothing$ or $V(T) = V(G)$. We know the root of the tree is in $V(T)$, thus $V(T) \neq 0 \to V(T) = V(G)$, so it's a spanning subgraph and by construction T is a tree. □

Definition 20.2. If T is a BFST rooted at r, for any vertex $x \in V(T)$, we define the **parent of** x to be the first neighbor of x before x in the queue.
The root r therefore has no parent. We write $pr(x)$ for the parent of x where $pr(r) = \varnothing$.

Definition 20.3. The **level of** x is $level(x) = level(pr(x)) + 1$ where $level(r) = 0$.

20.2 Shortest Path by Property of BFST

Assume G is a connected graph. Let T be a BFST of G, rooted at r.
For every edge $\{x, y\} \in E(G)$, $|level(x) - level(y)| \leq 1$. See course notes for details. We can use this property to find the **shortest path from** x **to** y.

Theorem 20.2. Let $x, y \in V(G)$ where G is a connected graph.
If T is a BFST rooted at x, then the unique path from x to y in T is the shortest path from x to y in G.
This is the path $y, pr(y), pr(pr(y)), pr^3(y), \ldots, x$ and is the length of $level(y)$.

Proof. We need to show that every path $v_0, v_1, \ldots, v_n$ with $v_0 = y$, $v_n = x$ has length $n \geq level(y)$. We have

$$\begin{aligned} n = \sum_{i=1}^{n} 1 \geq \sum_{i=1}^{n} &|level(v_i) - level(v_{i-1})| \\ &\geq |\sum_{i=1}^{n} level(v_i) - level(v_{i-1})| \\ &= |level(v_1) - level(v_0) + level(v_2) - level(v_1) + level(v_3) - level(v_3) + \ldots + level(v_n) - level(v_{n-1}) \\ &= |level(v_n) - level(v_0)| \\ &= |level(y) - level(x)| \\ &= level(y) \end{aligned}$$

□

Intuitive idea: Each step in the path can only go up one level at best.

21 June 26, 2017

21.1 Girth

Definition 21.1. If G is a graph with a cycle, then the **girth** of G is defined to be the *length of a shortest cycle.*

21.2 Shortest Cycle

How do we find the shortest cycle? We can use the BFST algorithm.

1. For each vertex $v \in V(G)$, construct a BFST T_v rooted at v
2. For every edge $f \in E(G) \setminus E(T_v)$, consider $T_v + f$ (we have a bunch of these). $T_v + f$ must contain a cycle. Out of all $T_v + f$, there exists our shortest cycle (with length girth).

21.3 Planar Graphs

Definition 21.2. A **planar embedding** of a graph G is a drawing of G in the *plane with no edges crossing.* That is a graph is **planar** if it has a planar embedding (the *drawing itself*).

1. **3-cube** is planar (can flatten the cube) with no edges crossing.
2. K_5: got stuck trying to find a planar embedding (does not mean it is not planar).

There are infinitely many ways to draw a graph, thus trying a couple and failing *does not* prove a graph is not planar.

21.4 Faces (of Planar Embeddings)

Definition 21.3. A planar embedding divides the plane into regions called **faces**. Note the region around the graph is also a face.

Note the following addition definitions:

- Two faces are said to be **adjacent** if they are incident with a common edge

- The **boundary of a face** is the subgraph consisting of vertices and edges incident with the face
- The **degree of a face** is

$$(\# \text{ of non-bridges in the boundary}) + 2(\# \text{ of bridges in the boundary})$$

Why 2 for bridges? In a planar embedding, note that:

- a bridge is incident with only one face
- a non-bridge is incident with two faces

Proof uses Jordan curve theorem.

21.5 Handshaking Lemma for Faces

Theorem 21.1. The total degree of faces is 2 times the number of edges or for a planar embedding with faces $f_1, \ldots, f_s$

$$2|E(G)| = \sum_{i=1}^{s} deg(f_i)$$

Proof. Each edge contributes:

- 1 to the degree of 2 different faces if the edge is not a bridge
- 2 to the degree of 1 face if the edge is a bridge

Either way the edge contributes 2 to the sum of the degrees of the faces. □

22 June 28, 2017

22.1 Alternate Definition of a Degree of Face

For **connected graphs**, the degree of a face is equal to the length of its boundary walk.
For example in a graph with 4 vertices, 3 connected in a triangle labeled $1, 2, 3$ and 1 vertex sticking out from vertex 2 on the triangle labeled 4, the outer face's boundary walk would be $1, 2, 4, 2, 3, 1$ of length 5 (number of commas), thus the outer face has degree 5.

22.2 Isomorphic Graphs and Degrees of Faces

Given two isomorphic graphs, the degrees of each individual face may differ. Regardless, the sum of of the degrees of faces is still the same (obviously from the handshaking lemma).
Can we determine the number of faces in a planar embedding from the graph? **Yes!**

22.3 Euler's Formula for Planar Graphs

Theorem 22.1. If P is a *connected* planar embedding with p vertices, q edges, and s faces then

$$p - q + s = 2$$

We will first prove the generalization

Theorem 22.2. For a graph with p vertices, q edges, s faces and c components

$$p - q + s = c + 1$$

Proof. By induction on q the number of edges
Base case: If P is a planar embedding with $q = 0$ edges then note $c = p$ (every vertex is a component by itself). Furthermore $s = 1$ (the whole plane is the only region/face). Thus $p - q + s = p + 1 = c + 1$.
Induction hypothesis: Assume $q \geq 1$ and the result is true for planar embeddings with $q - 1$ edges.
Inductive step: Let P be a planar embedding with q edges, p vertices, s faces, and c components. Let e be any edge of P. Consider $P - e$:

- If e is a bridge then $P - e$ has
  - p vertices
  - $q - 1$ edges
  - s faces (same face on both sides of a bridge)
  - $c + 1$ components (deleting a bridge creates 1 more component by theorem)

  Note $P - e$ is planar so by the inductive hypothesis

$$p - (q - 1) + s = (c + 1) + 1 \rightarrow p - q + s = c + 1$$

  as desired.

- If e is not a bridge then $P - e$ has
  - p vertices
  - $q - 1$ edges
  - $s - 1$ faces (non-bridge incident with 2 different faces which becomes one when e is deleted)
  - c components (not a bridge)

  By the induction hypothesis

$$p - (q + 1) + (s - 1) = c + 1 \rightarrow p - q + s = c + 1$$

  as desired.

□

22.4 Degree of Faces and Girth

Theorem 22.3. If P is a planar embedding with a cycle, then every face f of P has

$$deg(f) \geq girth(P)$$

Proof. **Sketch of proof**

$$\begin{aligned}\text{cycle} &\rightarrow \text{non-bridge edges}\\ &\rightarrow \text{more than one face}\\ &\rightarrow \text{no face is all of the plane}\\ &\rightarrow \text{boundary of every face must contain a cycle}\\ &\rightarrow \text{degree of face} \geq \text{length of that cycle} \geq \text{girth}\end{aligned}$$

□

23 June 30, 2017

23.1 Key Formulas for Planar Embeddings

Where $p = \#$ of vertices, $q = \#$ of edges, $s = \#$ of faces, and $f_1, \ldots, f_s$ are the faces:

1. $\sum_{v \in V(G)} deg(v) = 2q$ (Handshake Lemma for vertices; holds for any graph)
2. $\sum_{i=1}^{s} = deg(f_i) = 2q$ (Handshake Lemma for faces)
3. $p - q + s = 2$ (Euler's formula for connected graphs)
4. $deg(f_i) \geq girth(G)$ for all f_i

23.2 Proof of Non-Planar Graphs

We can use the key formula to prove that some graphs are not planar.

Example 23.1. Let's prove K_5 (complete graph with 5 vertices) is not planar.

Proof. Suppose to the contrary that we have a planar embedding of K_5. Therefore $p = 5$ and $q = \binom{5}{2} = 10$.
K_5 is connected so by Euler's formula $s = 2 + q - p = 7$. Let $f_1, \ldots, f_7$ be the faces. Notice that the smallest cycle has length 3, thus by (4) we have $deg(f_i) = 3$.
Handshake Lemma for faces says $2q = 20 = \sum deg(f_i) \geq 7 \cdot 3 = 21$. This is a contradiction, thus K_5 has no planar embedding. □

Exercise: Use the same technique for the Petersen graph, $K_{3,3}$.

Example 23.2. $K_{3,3}$ is not planar.

Proof. Note $p = 6, q = 3 \cdot 3 = 9, s = 2 + q - p = 5$.
Note the girth is 4, so $deg(f_i) \geq 4$.
$2q = 18 = \sum deg(f_i) \geq 5 \cdot 4 = 20$ which is a contradiction. □

If no contradiction is extracted, then G may or may not be planar (indeterminate).

23.3 Edge Bound by Vertices for Planar Embeddings

Theorem 23.1. If P is a *planar embedding*, with $p \geq 3$ vertices and q edges, then

$$q \leq 3p - 6$$

The converse is not necessarily true (cannot prove a graph is planar using this).
This theorem is often used in contrapositive: that is if $q > 3p - 6$, then the graph is not planar.

Proof. We will prove this for connected graphs.

Proof. There are two cases:

Case 1 If P does not have a cycle, then it's a tree so $q = p - 1$. Since $p \geq 3$, then $q = p - 1 \leq 3p - 6$.

Case 2 If P has a cycle then P has $s = 2 + q - p$ faces $f_1, \ldots, f_s$. Note that $deg(f_i) \geq girth(P) \geq 3$ (minimum cycle is 3). So $2q = \sum deg(f_i) \geq 3s$, where $s \leq \frac{2q}{3}$.

Thus $\frac{2q}{3} \leq 2 + q - p \rightarrow q \leq 3p - 6$.

□

□

23.4 Edge Bound by Vertices for Bipartite Planar Embeddings

Theorem 23.2. If P is a bipartite planar embedding with $p \geq 4$ vertices and q edges, then

$$q \leq 2p - 4$$

Proof. Proof is similar to above, but $girth(P) \geq 4$ (bipartite min cycle). □

23.5 Platonic Solids

3-d objects where there is a planar embedding (e.g. tetrahedral, cube, etc.). There are only 5 platonic solids. Why?

Proof. Note that the planar embeddings of Platonic solids have the following properties:

- Every vertex has same degree d
- Every faces has same degree d^*

We can deduce that

1. $pd = 2q$ (Handshake Lemma for vertices)
2. $sd^* = 2q$ (Handshake Lemma for faces)
3. $p - q + s = 2$ (Euler's Formula)

From (1), note that $q = \frac{dp}{2}$ From (2) we get $s = \frac{2q}{d^*} = \frac{pd}{d^*}$.
Thus from (3) we get

$$p - \frac{dp}{2} + \frac{pd}{d^*} = 2$$
$$\rightarrow p - dp(\frac{1}{2} - \frac{1}{d^*}) = 2$$
$$\rightarrow p - 2 = dp(\frac{1}{2} - \frac{1}{d^*})$$
$$\rightarrow d = \frac{p-2}{p}(\frac{1}{2} - \frac{1}{d^*})^{-1}$$

Note $\frac{p-2}{p} \leq 1$, thus

$$d < (\frac{1}{2} - \frac{1}{d^*})^{-1}$$
$$\rightarrow (d-2)(d^*-2) < 4$$

Note that $d \geq 2$ (if you have more than 1 vertex, it must have at least degree 2) and $d^* \geq 3$.
Thus there are 5 possibilities for (d, d^*) that are 3d: $(3,3),(3,4),(4,3),(5,3),(3,5)$. Note that $(2,n)$ for any n is a n-sided 2-d polygon, thus it is not a platonic solid. Therefore there are 5 Platonic solids. □

24 July 5, 2017

24.1 Prove That a Graph is Planar

Note we can **NEVER use the properties before to prove a graph is planar**. That means there are some graphs that are non-planar for which we cannot prove (currently).

24.2 Kuratowski's Theorem

Note if we have a K_5 graph (graph with 5 vertices of degree 4, pentagon with star in the centre) and we add vertices (internal vertices) on the edges adjacent to the 5 vertices (BUT not on the intersection of the internal edges), instead of an edge joining the 5 vertices, we have a path. Note that the paths have *no internal vertices in common* (if we do not put a vertex on two simultaneous edges).
This is called an **edge subdivision** of K_5. (We are dividing the edges of K_5 into paths of length ≥ 2) It is not planar.
Note: If we did somehow end up putting a vertex on the intersection of two internal edges, then we will get a planar graph!
More formally

Theorem 24.1. A graph is non-planar if and only if it has a subgraph that is isomorphic to an edge subdivision of K_5 or $K_{3,3}$.

Proof. **Backwards direction:**

- K_5 and $K_{3,3}$ are non-planar
- Edge subdivisions of K_5 and $K_{3,3}$ are non-planar (why?)
- Any graph containing a non-planar subgraph is non-planar (Similarly, subgraph of a planar graph is planar)

Forwards direction:
Hard. □

24.3 Kuratowski Example with Petersen Graph

Note the Petersen graph has vertices with degrees at most 3. This tells us we are looking for a subgraph of $K_{3,3}$ instead of K_5 (which requires degree 4).
We do this by finding 3 partition A and 3 partition B vertices then find paths/edges between the A and B vertices. Note in the above Petersen graph, we can let $e, f, b \in A$ and $a, i, h \in B$. We now describe the unique paths (with no common internal vertices) from each vertices in A to B.
The paths are (from the Petersen graph)

$$a, e$$
$$a, f$$
$$a, b$$
$$i, f$$
$$i, g, b$$
$$i, d, e$$
$$h, f$$
$$h, j, e$$
$$h, c, b$$

where the middle vertices are the unique internal vertices (that subdivided this graph). Thus the Petersen graph is not planar.

25 July 7, 2017

25.1 Recap on Planarity Checks

To check if a graph is planar or not:

1. Can you draw a planar embedding? If yes, then it's planar. Otherwise...
2. Is Euler's formula violated for *connected* graphs? If yes, then it's not planar. Otherwise...
3. For *connected* bipartite graphs, $p \geq 3$ vertices and q edges, is $q \leq 2p - 4$ violated? If yes, then it's not planar. Otherwise...
4. For *connected* graphs, $p \geq 3$ vertices and q edges, is $q \leq 3p - 6$ violated? If yes, then it's not planar. Otherwise...
5. Can you find a subdivision K_5 or $K_{3,3}$? If yes, then it's not planar.

25.2 Colouring Graphs

Definition 25.1. A **k-colouring** of a graph G is a function from $V(G)$ to a set of size k (elements are called colours) so adjacent vertices always have different colours. A graph with a k-colouring is called **k-colourable**.

25.3 2-Colourable and Bipartite

Theorem 25.1. A graph is 2-colourable if and only if it is bipartite.

Proof. If a graph is bipartite, then we can colour all vertices in A one colour and all vertices in B another colour. Note all edges include one vertex from each set and join vertices of different colours hence it is 2-colourable.
If a graph is 2-colourable, then we can assign vertices of one colour as partition A and the other as B. By definition, all edges join vertices of different colours hence it is a bipartition. □

25.4 n-Colourable and K_n

Theorem 25.2. A graph K_n is n-colourable, and not k-colourable for $k < n$.

Proof. A graph with n vertices is n-colourable (each vertex has a different colour thus there can be no adjcaent vertices with the same colour).
Note graph K_n is not k-colourable for any $k < n$ since that would result in two arbitrary vertices having the same color. Since in a K_n a given vertex is adjacent to all other vertices, there will be two adjcaent vertices with the same colour. □

25.5 6-Colourable for Every Planar Graph

We first must prove an important corollary from $q \leq 3p - 6$.

Theorem 25.3. Every planar graph has *a vertex* of at most degree five.

Proof. This is obviously true for $|V(G) \leq 2$. for $p = |V(G)| \geq 3$ and $q = |E(G)|$ we have

$$\begin{aligned} & q \leq 3p - 6 \\ \iff & \frac{2q}{p} \leq 6 - \frac{12}{p} \end{aligned}$$

This shows that the average degree of a vertex is less than six. Thus there exists a vertex of degree at most five. □

We can use this to show:

Theorem 25.4. Every planar graph is 6-colourable.

Proof. Proof by induction on p vertices: note for graphs with $p = 1$ or one vertex, it can obviously be 6-colourable so this is true.
Induction hypothesis assumes that all planar graphs on $p \leq k$ vertices are 6-colourable, $k \geq 1$.
We have a graph G with $p = k + 1$ vertices. We know from the previous thoerem that a planar graph has one vertex v of degree at most five. We take $G - v$ and colour it with 6 colours from the induction hypothesis. Then we introduce v back in that is adjacent with at most 5 colours. We can thus colour it with one of 6 colours that is not the same as any of its 5 adjcaent colours hence this graph G is 6-colourable by induction. □

26 July 10, 2017

26.1 5-Colourable for Every Planar Graph

Theorem 26.1. Every planar graph is 5-colourable.
We will prove this by induction on $p = |V(G)|$.

Proof. **Case 1:** If there is a vertex of degree at most 4, we can use the same argument we used in 6-colourable proof.
Case 2: There is no vertex of degree at most 4. By Corollary 7.5.5, there is a vertex of degree 5, say v. Main idea: Can we ensure that $G - v$ have a 5-colouring in which two of u_1, u_2, u_3, u_4, u_5 (neighbors) use the same colour? Note that if there is an edge between every vertex u_i, u_j where $i \neq j$ then they all must be different colours. However, this cannot happen since G will have a K_5 subgraph. So there exists u_i, u_j, $i \neq j$ such that u_i, u_j are not neighbors. Let's say they are u_2 and u_4.
In our original graph G, let e_2, e_4 be the edges between v and u_2, u_4 respectively. We contract edges e_2, e_4 which would move the (4 each) incident edges of e_2 and e_4 to a new vertex w (replaced v). w is incident to u_1, u_3, u_5 still. Note that this new graph H has 2 fewer vertices and is still planar thus by the induction hypothesis H is 5-colourable.
So between u_1, u_3, u_5, w we have 4 colours. When we expand the edges again, we can assign u_2, u_4 the colour of w (colouring still holds since w was incident to the edges of u_2, u_4) and v the 5th colour that was not used amongst u_1, u_3, u_5, w in H. □

26.2 Contracting Edges

Note in the 5-colourable proof we used edge contraction. Formally, G/e is G with edge $e = uv$ contracted. We identify the two vertices u and v and combine them into one vertex (combining incident edges), delete edge e, then remove any parallel edges.

26.3 Matching

Definition 26.1. A **matching** M in a graph $G = (V, E)$ is a subset of edge set $M \subseteq E$ such that no two edges share a vertex. That is if M is a matching then for every vertex v, either $|M \cap \delta(v)| = 1$ or $|M \cap \delta(v)| = 0$.

Note that every graph has a matching since the empty set is a matching and $\{e\}$ is a matching for any $e \in E$. Therefore all matchings M of edges e is bounded by

$$|M| \leq \frac{|V|}{2}$$

26.4 Maximum vs Maximal Matching

Definition 26.2. A **maximum matching** is a matching that uses as many edges as possible (out of all matchings).

Definition 26.3. A **maximal matching** is a matching that cannot be made any bigger by adding an edge (so a maximal matching may not be a maximum matching, only maximal given the edges in the matching).

27 July 12, 2017

27.1 Perfect Matching

A perfect matching is a matching where every vertex is matched.
Some observations about perfect matchings:

(i) A graph must have even number of vertices to have a perfect matching (thus a graph with odd number of vertices has no perfect matching)

(ii) A graph with even number of vertices DOES NOT imply there exists a perfect matching (e.g. graph with two vertices and no edges)

(iii) The Petersen graph has multiple perfect matchings

(iv) If a graph has a perfect matching, then the maximum matchings are the same as the perfect matchings

27.2 Applications of Matchings

Example 27.1. Job applications with set of applicants $A = \{a, b, c\}$ and set of jobs/positions $J = \{1, 2, 3, 4\}$. Let's say:

- a is qualified for $1, 2$
- b is qualified for $2, 4$
- c is qualified for $1, 3, 4$

Our goal is to give as many applicants a job and (a) each applicant gets at most one position and (b) each position is assigned at most one applicant.
This is isomorphic to finding a maximum matching in the associated bipartite graph.

We can apply the same concept to numerous important problems (organ donation).

27.3 Finding a Larger Matching

Easy Strategy Given a graph G and a matching M, we can try to add more edges to the matching M.

This however does not work for maximal matchings.

Sophisticated Strategy Same we have a graph where there is a longest path (that includes all vertices) $1, 8, 2, 3, 7, 6, 4, 5$ (length 7 with 8 vertices).

Let's say we initially started with the maximal matching M with 3 edges $\{8, 2\}$, $\{3, 7\}$, and $\{6, 4\}$. Note 1 and 5 are **M-unmatched**.

The path itself is called a **M-augmenting path**.

What we can do is invert the edges in our matching (throw away the 3 edges and take $\{1, 8\}$, $\{2, 3\}$, $\{7, 6\}$, and $\{4, 5\}$). Thus we get a maximum and perfect matching with 4 edges.

27.4 M-Alternating and M-Augmenting Path

Definition 27.1. Let G be a graph and let M be a matching. Let $P = e_1, e_2, \ldots, e_k$ (ordered sequence of edges). Then P is an **M-alternating path** if $\forall i \in \{1, \ldots, k-1\}$ exactly one of e_i and e_{i+1} is in M.

Definition 27.2. An **M-augmenting** uv**-path** P is an *M-alternating path* with the additional property that both ends u and v are *M-unmatched*.

Note that if there is an *M-augmenting path* P then we can obtain a bigger matching M^* as follows:

- We have two sets of edges: M and P. We can draw a venn diagram for this.
- Note the intersect of M and P are all matching edges on the path.
- $M^* = M\Delta P = M \cup P \setminus (M \cap P)$: we take all edges in M outside of P (leave as), remove all edges that are currently on the path P and in the matching M ($M \cap P$), and we take all edges in the path not in the matching (the inverting the edges part).

In our above example, note that $M \subset P$ (all edges in M were on the path). This may not be the case for all examples! Furthermore, $|M^*| = |M| + 1$ (we added one additional edge to our matching).

Lemma 27.1. A matching M with an M-augmenting path is NOT a maximum matching.

Note **M-augmenting paths** in a matching M in a graph G need not be maximal, as long as the end points are unmatched in the entire graph.

28 July 14, 2017

28.1 Finding a Maximum Matching in Any Graph

1. Start with $M = \varnothing$ (one may instead start with a maximal matching)
2. If M is a max matching, then STOP (done)
3. Otherwise find an M-augmenting path P, then update $M = M\Delta P$. Go to step 2.

There are two difficulties with this algorithm:

1. How do you know that the current matching M is maximum?
2. How do you find an M-augmenting path P?

We will answer this for bipartite graphs.

28.2 Cover Subset of Vertices

Definition 28.1. In a graph $G = (V, E)$ in a subset S of vertices ($S \subseteq V$) is a **cover** if every edge of G has at least one of its ends in S (that is, S hits/covers every edge).

Note a cover can include full edges (e.g. an edge has both ends in S).

28.3 Minimum Cover

Given a graph G, what is the size of the smallest cover or minimum cover?
Note that it does not necessarily equal the size of the maximum matching (e.g. K_3 has a maximum matching of size 1 but a minimum cover of size 2).
Is it true that for any graph G that the size of a maximum matching is $\leq$ size of a minimum cover? We can reform this question: is $|M| \leq |C|$ for *any matching* M for any cover C of a given graph G?
Yes this is true.

Example 28.1. Given a matching of size 3 $M = \{\{1,2\},\{3,6\},\{4,5\}$, to cover each edge of the matching, we need to select at least 1 endpoint from each edge in M. This is because a matching only matches a vertex once, thus no vertex is repeated in the matching. We thus must choose a vertex from every edge to cover every edge M for a total of at least 3 vertices thus $|C| \geq 3$.

Lemma 28.1. If M is a matching of G and C is a cover of G then $|M| \leq |C|$.

Lemma 28.2. Suppose that for a graph G you have a matching M and a cover C such that $|M| = |C|$ (equality). Then M is a maximum matching and C is a minimum cover (The converse is not necessarily true as shown before with K_3).

28.4 Konig's Theorem

Theorem 28.1. In a bipartite graph, the size of a maximum matching is equivalent to the size of a minimum cover.

29 July 17, 2017

29.1 Proof of Konig's Theorem

Proof. Idea behind the proof:
We have:

- a bipartition (A, B)
- matching with saturated (touch matching edge) and unsaturated vertices
- Reachable/unreachable vertices

Reachable/Unreachable:

- Let X_0 be the set of all unsaturated vertices in A. Every vertex in X_0 is reachable.
- If y is a vertex such that there exists an alternating path from a vertex $x \in X_0$ to y, then y is reachable
- All other vertices is unreachable

Example 29.1. In the above figure, let the LHS be A and the RHS be B. Let $M = \{\{a,z\},\{b,x\}\}$.
Note $X_0 = \{c\}$ (unsaturated in A). Thus c is reachable. Note c, x and c, y are alternating paths of length 1 with 1 unmatched edge, thus x and y are reachable.
Note c, x, b is an alternating path of length 2 (unmatched, matched edge) thus b is also reachable. c, x, b, w is an alternating path of length 3 (unmatched, matched, unmatched) thus w is reachable.
Note there is no alternating path to a, thus all vertices are reachable except for a.

Lemma 1:

Lemma 29.1. Let X be the set of reachable vertices in A, and Y set of reachable vertices in B.
If $u \in X$ and $e = \{u, v\}$ is an edge of G, then $v \in Y$.
That is there is no edge between the reachables in A to the unreachable in B (but there may be edges between every other reachable/unreachable groups in A and B, obviously not between groups in the same part).
By definition, if u is a reachable vertex, there is an alternating path $P(u)$ from a vertex in X_0 to u. **Note:** if $u \in X$ (last vertex) then the last edge in $P(u)$ must be in M.
Proof of lemma:

Proof. Assuming $u \in X$ the last vertex in $P(u)$, there are two cases:

1. If $e \in M$ (in matching), then e must be the last edge in $P(u)$.

   Why? The last edge in $P(u)$ is in M and incident with u, and since this is a matching only one matching edge is incident with a saturated u.

2. If $e \notin M$, then consider $P(u)$. We want to show that $v \in Y$.

   If $P(u)$ reaches v before reaching u, then v is reachable or $v \in Y$ (since $u \in X \subseteq A$).

   If v is in this path then v is reachable (done), so assume it isn't.

   Write $P(u) = x, \dots, u$ where $x \in X_0$. Then $x, \dots, u, v$ is a path from x to v.

   Furthermore, this is alternating because $e \notin M$ and the last edge in $P(u)$ is in M. So v is reachable and thus $v \in Y$ (since $u \in X \subseteq A$).

□

Lemma 2:

Lemma 29.2. If $v \in Y$ and $e = \{u, v\} \in M$, then $u \in X$.
That is all matching edges must be between saturated vertices in the reachable parts, or between saturated vertices in the unreachable parts.

Proof. Similar to previous lemma. See course notes. □

Lemma 29.3. $C = Y \cup (A \setminus X)$ is a cover (reachable parts of B (Y) and unreachable parts of A ($A \setminus X$) form a cover.

Proof. See diagram. This is an immediate consequence of Lemma 1: G has no edges joining a vertex in X and a vertex in $B \setminus Y$. □

Lemma 29.4. Let M be any matching. Is there an unsaturated vertex in Y (or B reachable)?

Proof. Two possibilities:

Yes Suppose $v \in Y$ is an unsaturated vertex. Since $v \in Y$, v is reachable, so there exists an alternating path $P(v)$ starting at an unsaturated vertex in X (reachable of A) and ending at v.

$P(v)$ is an augmenting path since both ends are unsaturated. Thus M cannot be a maximum matching.

No Every vertex in C (our cover which is Y and $A \setminus X$, saturated parts of A) is saturated. Also since there are no matching edges joining a vertex in Y to a vertex in $A \setminus X$ (by Lemma 2), every edge in M is incident with a unique vertex in C (see diagram)

We have a one-to-one correspondence between M and C, therefore $|C| = |M|$, and as proven last week M is a maximum matching and C is a minimum cover.

Finally, suppose now that M is a maximum matching. As we just showed, this implies there is no unsaturated vertex in Y. Therefore $C = Y \cup (A \setminus X)$ is a min cover of the same size as M.

□

□

30 July 19, 2017

30.1 Bipartite Matching Algorithm

We modify breadth first search with some changes:

- Queue begins with all vertices in X_0 (unsaturated vertices in A)
- Note instead of a tree, we get a forest
- If *active vertex* is in A, only include non-matching edges (so we can only follow blue edges out of A)
- If *active vertex* is in B, only include matching edges (we can only follow red edges out of B)

Example 30.1. In the given graph, we first start with $X_0 = \{1, 4\}$. Push 1 and 4 into queue. 1 is active.
Note every time we push in a vertex, we draw out the corresponding edge to generate a tree.
Push in 8, done with 1. Our queue is $4, 8$.
Push in 10, done with 4. Our queue is $8, 10$.
Push in 3, done with 8. Our queue is $10, 3$.
Push in 5, done with 10. Our queue is $3, 5$.
Push in 6 and 9, done with 3. Our queue is $5, 6, 9$.
There are no adjacent vertices to 5 left. Similarly for 6 and 9. Thus we are done.
We end up with two trees that start at 1 and 4.
Let X be $\{\text{vertices in forest}\} \cap A$ and Y be $\{\text{vertices in forest}\} \cap B$. Thus

$$X = \{1, 3, 4, 5\}$$
$$Y = \{6, 8, 9, 10\}$$

We can thus draw out our diagram with the four groups with the vertices and edges.
Note there are unsaturated vertices in Y thus there are augmenting paths and our current matching isn't maximum.
That is $1, 8, 3, 6$ and $1, 8, 3, 9$ are augmenting paths. We flip the edges in one of these paths and repeat.
Note $X_0 = \{4\}$. Begin our queue with 4.
Push in $8, 10$. Queue is $8, 10$.
8 is active, push in 1. Queue is $10, 1$.
10 is active, push in 5. Queue is $1, 5$.
No available vertices adjacent to either 1 or 5. We are done.
Note that there are no unsaturated vertices in Y in the configuration $X = \{1, 4, 5\}$ and $Y = \{8, 10\}$.
Thus we have a maximum matching $\{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{5, 10\}\}$ where our min cover $C = Y \cup (A \setminus X) = \{2, 3, 8, 10\}$.

31 July 21, 2017

31.1 Modelling Problems as Bipartite Graphs

Note we can model problems like assigning workers to shifts as bipartite graphs (A is workers and B is shifts).
If a worker wants to work 2 shifts, then we can simply create a copy of it as a second vertex in A, and we can generalize this to n shifts or m workers per shift.
We can even model more complicated problems: if there are a group of students each of whom represent a certain number of clubs, how can we form a matching such that as many clubs are represented for a conference but one student cannot represent two clubs at the same time?

31.2 Hall's "Marriage" Thoerem

Let G be a bipartite graph with bipartition (A, B). When is it possible to find a matching that saturates every vertex in A?
Note:

- If $|A| > |B|$, then obviously this is not possible
- If $D \subseteq A$ (D is any subset of A), then we must have $|D| \leq |N(D)|$ where $N(D) = \{v \in B : v$ adjacent to same vertex in Otherwise we cannot match everyone in D anyways and thus every vertex in A cannot be matched.

Hall's Theorem:

Theorem 31.1. Let G be a bipartite graph with bipartition (A, B). There exists a matching saturating every vertex in A *if and only if* for every subset $D \subseteq A$ we have $|D| \leq |N(D)|$.

Proof. **Forwards direction:**
We showed this by showing that to saturate every vertex in A, there must be enough neighbours in B for every such subset $D \subseteq A$.
Backwards direction:
We need to show that if a maximum matching M have size $|M| < |A|$, then there exists a set D such that $|N(D)| < |D|$ (contrapositive).
Suppose $|M| < |A|$. Suppose M is a maximum matching.
Take $D = X$ (X in the construction from Konig's theorem, the reachable vertices in A). Then since $|M| < |A|$ there is an unsaturated vertex in X.
Since M is a maximum matching there is no unsaturated vertex in Y, thus $|X| > |Y|$ (if you look at Konig's theorem diagram).
Also $N(X) \leq Y$ (in fact, $N(X) = Y$ from the Konig diagram).
Thus $|N(X)| < |X|$ as desired. □

32 July 24, 2017

32.1 K-regular Bipartite Graph have Perfect Matching

Theorem 32.1. If G is a k-regular bipartite graph ($k \geq 1$) then G has a perfect matching.

Proof. Let (A, B) be the bipartition of G.

Lemma 32.1. $|A| = |B|$.
Proof of lemma:

Proof. We can show this by counting in two ways:

- Every edge is incident with exactly one vertex in A thus

$$q = \sum_{v \in A} deg(v) = k|A|$$

- Similarly for vertices in B

$$q = \sum_{v \in B} deg(v) = k|B|$$

thus $k|A| = k|B| \to |A| = |B|$. □

Let $D \subseteq A$. To show there is a perfect matching, we first show that $|N(D)| \geq |D|$.
Count edges incident with a vertex in D: Let

$$Q = \{e \in E(G) : e \text{ is incident with a vertex in } D\}$$

Note that

$$|Q| = \sum_{v \in D} deg(v) = k|D|$$

Now consider what happens with B: Every edge in Q is incident with a vertex in $N(D)$. But there are 0 or more edges not in Q that are incident with a vertex in $N(D)$. Thus

$$|Q| \leq \sum_{v \in N(D)} deg(v) = k|N(D)|$$

Therefore we have

$$k|D| \leq k|N(D)|$$

so $|N(D)| \geq |D|$.
Since this is true for every set $D \subseteq A$, by Hall's Theorem G has a matching that saturates every vertex in A.
Since $|A| = |B|$, this is a perfect matching. □

32.2 Bipartite Graph Example Problem

Example 32.1. Let there be a deck of 36 cards labelled 1-8.
There are i cards labelled i (1 card labelled 1, 2 cards labelled 2, etc.)
Deal the deck into 6 piles of 6 cards. Show that it's possible to pick one card from each pile such that there are no two cards with the same label.
Solution: Create a bipartite graph with bipartite (A, B) such that

$$A = \{p_1, p_2, \ldots, p_6\} \qquad \text{set of piles}$$
$$B = \{1, 2, 3, 4, \ldots, 8\}$$

There is an edge from i to p_j if there is a card labelled i in pile p_j.
We want to show that there is a matching that saturates every vertex in A (every pile has one unique label associated with it).
Let $D \subseteq A$ (subset of piles).
Consider cards in piles from D. How many different labels must we have?
Let $d = |D|$. We can do cases on d (number of piles in subset) to show that $|N(D)| \geq |D|$:

$d = 1$ We have 6 cards (a given pile has 6 cards), thus we must have at least 1 label.

$d = 2$ We have 12 cards in total. Note there is at most 8 cards of one label, thus we must have at least 2 labels.

$d = 3$ We have 18 cards in total. There is at most 8 cards of 8 and 7 cards of 7. Thus we must have at least 3 labels.

$d = 4$ 24 cards in total. At most $8 + 7 + 6 = 21$ cards of labels 8, 7 and 6. We must have at least 4 labels.

Similarly for $d = 5, 6$. Thus in all cases, number of labels is $|N(D)|$ and we have $|N(D)| \geq |D|$.
By Hall's theorem, every vertex in A can be saturated for a given matching.

32.3 Final Exam Notes

There will not be questions on:

- Section 2.8 - Recursive Decompositions of Binary Strings
- Section 3.4 - Nonhomogeneous Recurrence Relations
- Section 7.4 - Platonic Solids

32.4 Classes of Problems

- Extreme structures (e.g. longest path, shortest walk, maximum matching, etc.):

  We can use longest path to deduce a cycle by showing that the vertices have all degree at least 2 (thus we do not ever have to backtrack).

  We can use shortest walk to deduce a shortest path (by taking out loops or extract out a part of the walk that has a repeated vertex).

- Induction (use only when the question forces you to):

  - delete an edge
  - delete a vertex
  - delete a cycle

  This is useful if a deletion of the above *preserves the hypothesis*. So this is useful:

  - planar graphs: deleting a vertex/edge/cycle still maintains planarity
  - trees: deleting a leaf (some vertex + edge) still maintains tree. Deleting an edge creates two trees.

  A contrary example is for k-regular graphs: induction on vertices is usually a bad idea

- Subgraphs are graphs (e.g. show that certain properties do/don't hold for a given component)

- Use only what's explicitly mentioned (e.g. only focus on edges explicitly mentioned and ignore rest of graph, for example edges of explicitly mentioned matchings)

Figure 14.3: A Petersen graph

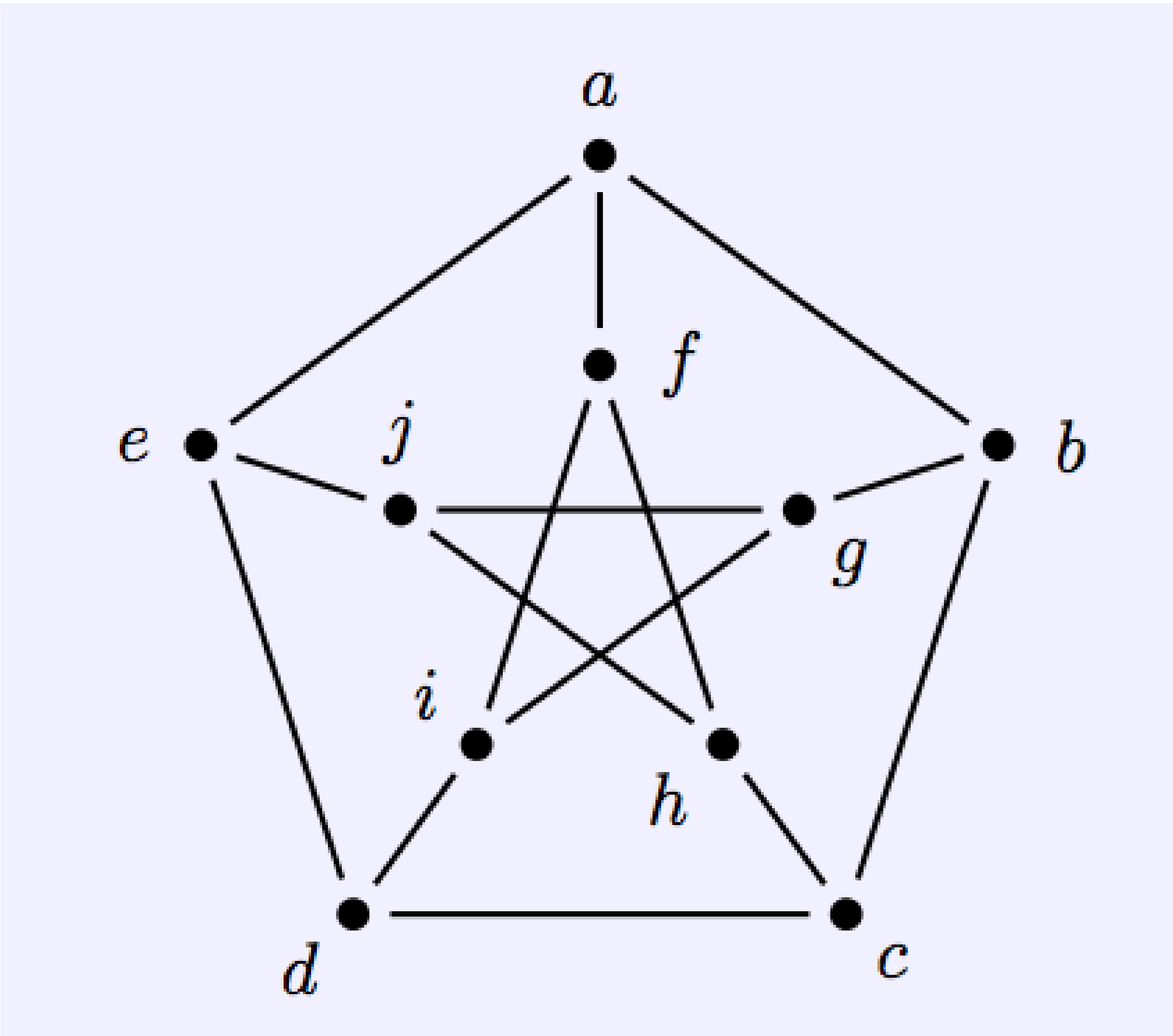


Figure 14.4: Bipartite graphs

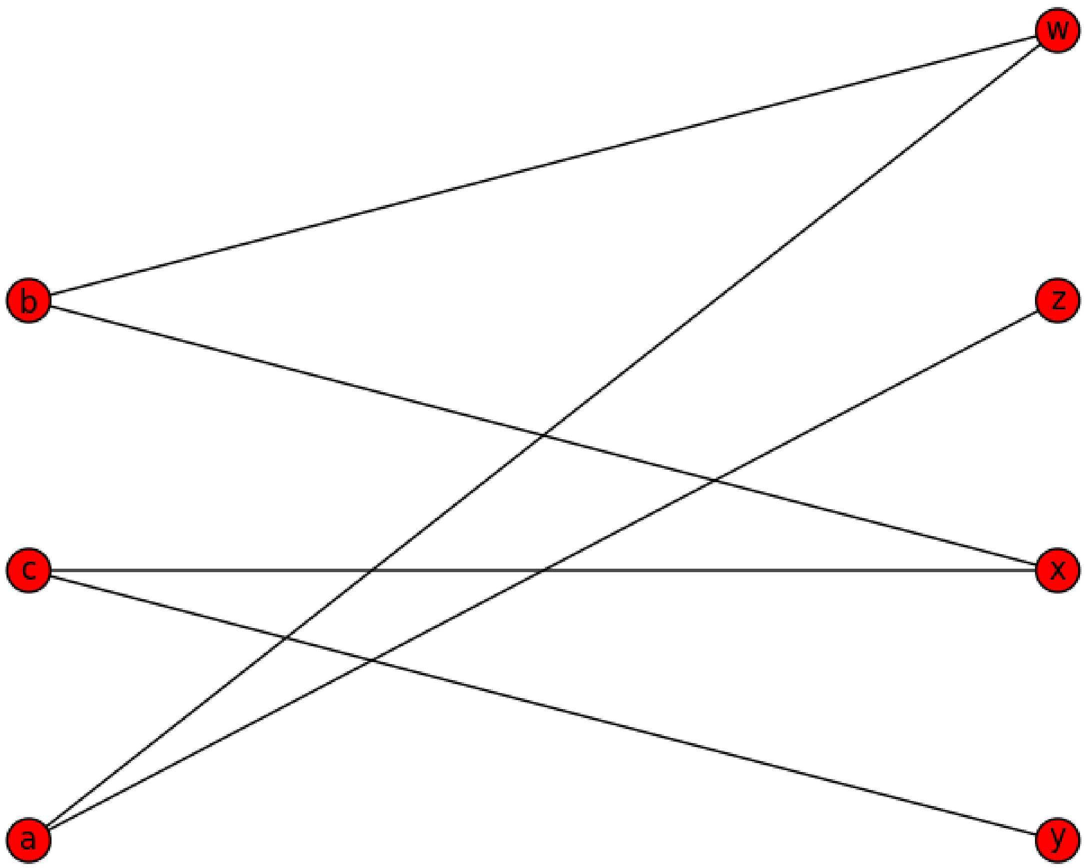


Figure 14.5: Complete bipartite graphs $K_{m,n}$

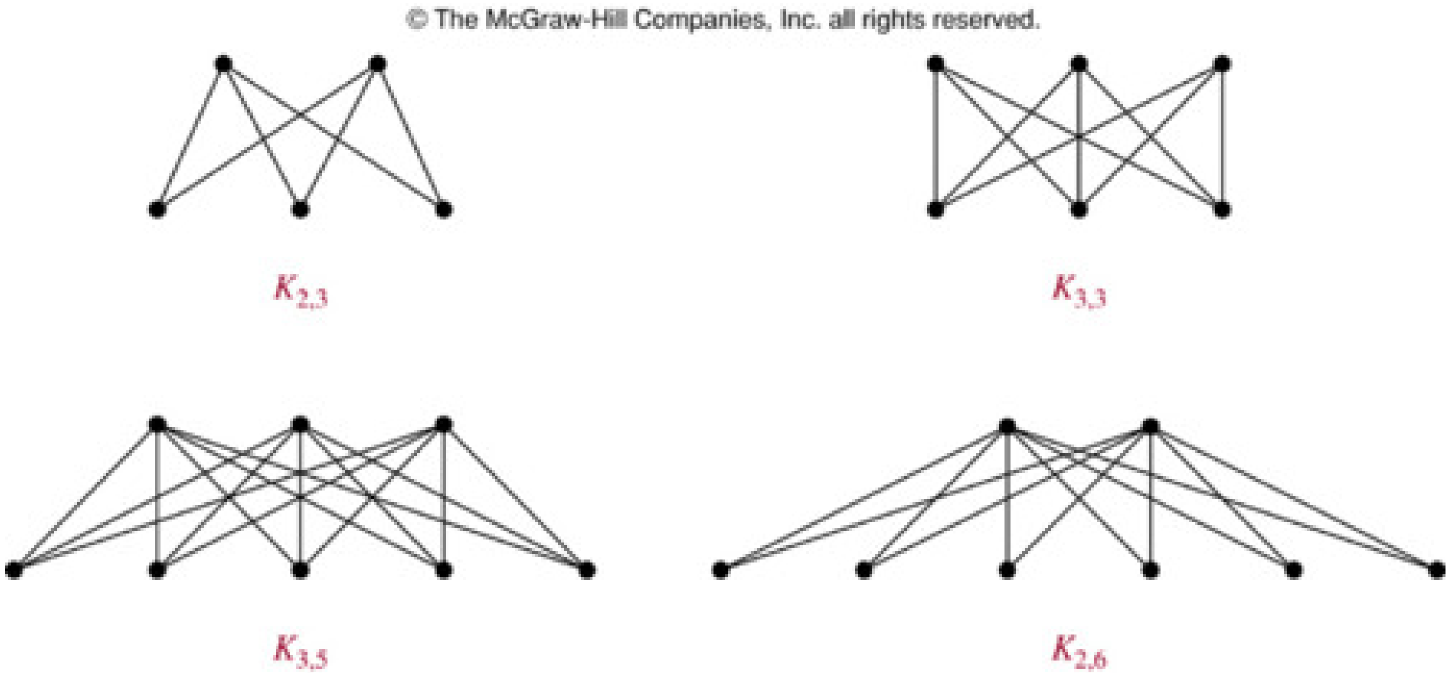


Figure 14.6: N-cube graphs Q_n with 2^n vertices.

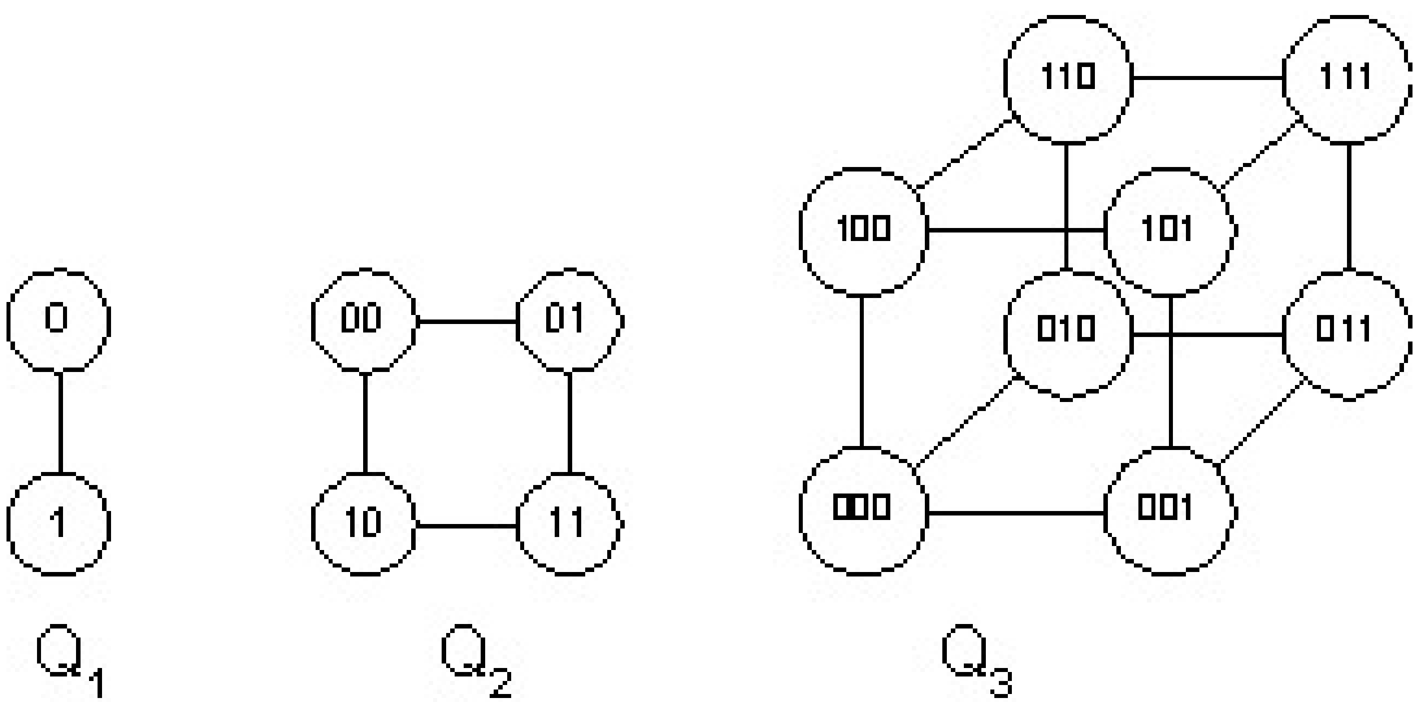


Figure 14.7: K_4 graph drawn as planar

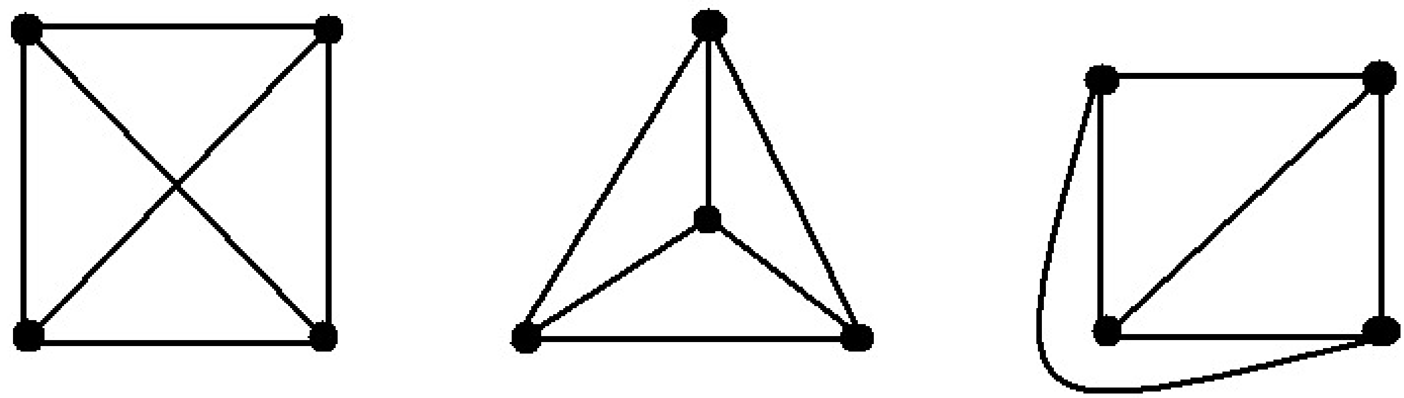


Figure 16.1: A Path Graph

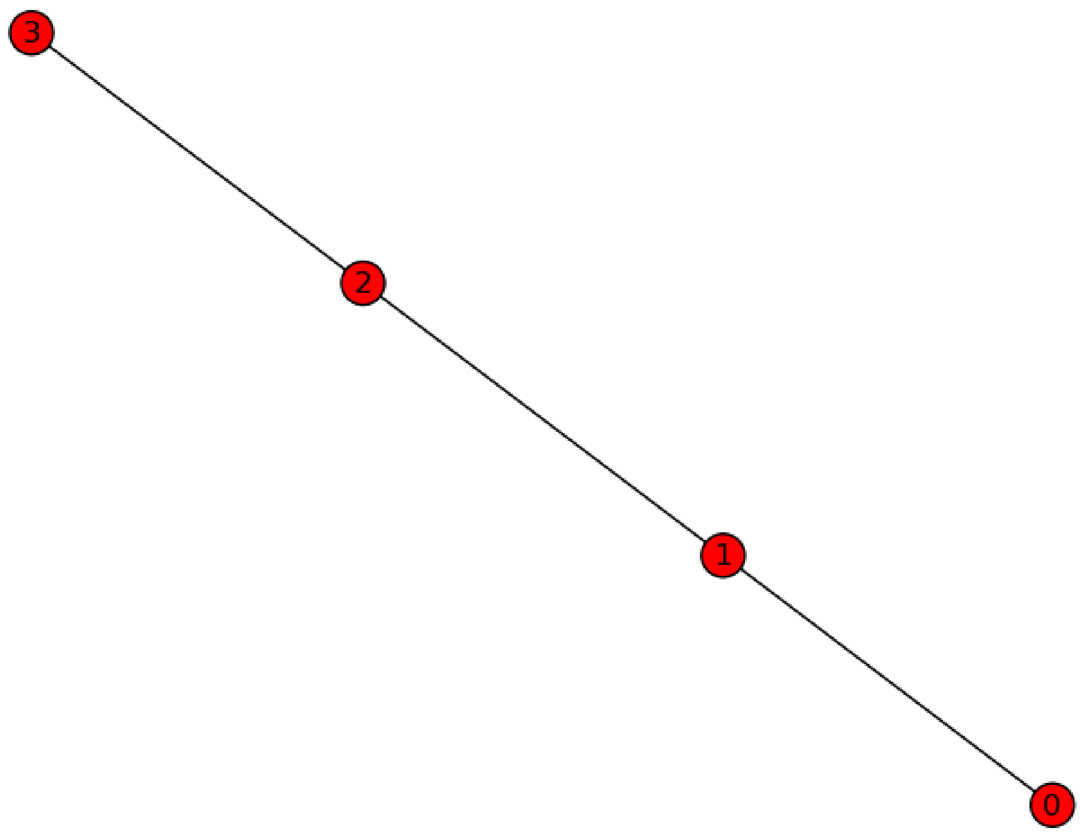


Figure 24.1: A Petersen graph

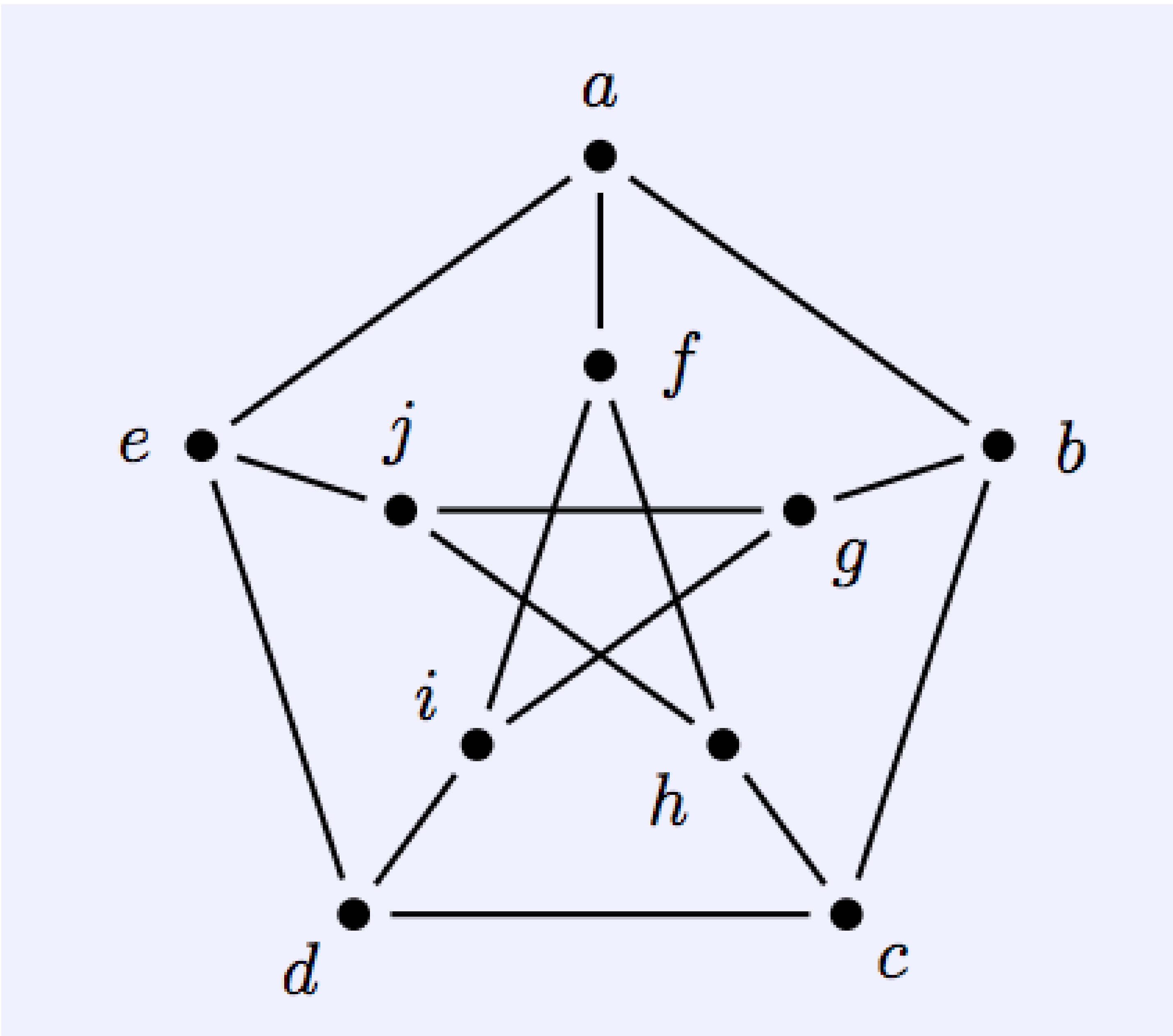


Figure 26.1: Maximum and maximal matching for a given graph.

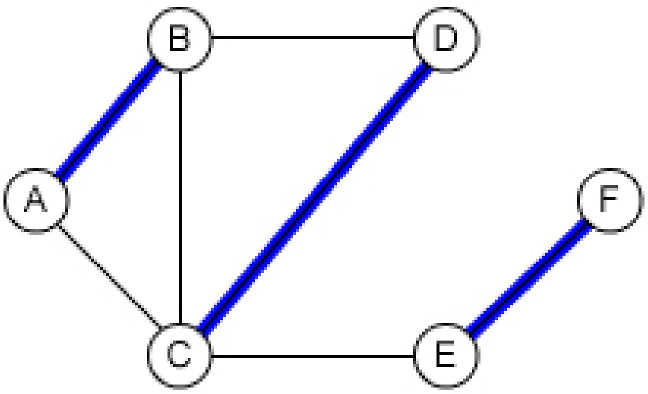


Figure 29.1: Example bipartite graph